REPRESENTATION THEORY OF SYMMETRIC GROUPS

WIJAYA SURYA

(B.Sc.(Hons;Cum Laude), Bandung Institute of Technology)

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It is proper for you, Kalamas, to doubt, to be uncertain; uncertainty has arisen in you about what is doubtful.
Come, Kalamas. Do not go upon what has been acquired by repeated hearing; nor upon tradition; nor upon rumor; nor upon what is in a scripture; nor upon surmise; nor upon an axiom; nor upon specious reasoning; nor upon a bias towards a notion that has been pondered over; nor upon another's seeming ability; nor upon the consideration, 'The monk is our teacher.'
Kalamas, when you yourselves know: 'These things are bad; these things are blamable; these things are censured by the wise; undertaken and observed, these things lead to harm and ill,' abandon them.
Kalamas, when you yourselves know: "These things are good; these things are not blamable; these things are praised by the wise; undertaken and observed, these things lead to benefit and happiness," enter on and abide in them.'

The Buddha

Dedicated to my dearest parents, in this life time and in my previous life time.

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Contents

Summary

In characteristic zero, the Specht modules S^λ give a complete set of mutually non-isomorphic simple modules for $k\mathfrak{S}_n$ as λ runs through the set of partitions of n. In positive characteristic p, the Specht modules S^λ have a simple head D^λ, provided λ is p-regular. The set of such simple heads is a complete list of mutually non-isomorphic simple modules of $k\mathfrak{S}_n$.

In chapter two, we do a survey of the general representation theory of the symmetric group. Our main resources for this chapter is [7].

In chapter three, we summarize some works done by Scopes [29, 30, 31], Martin and Russell [22, 23], and Tan [32, 33, 34] on blocks of small defect, which are related to our main aim.

In Chapter four, we discuss some methods to calculate the decomposition numbers. We calculate some decomposition numbers of blocks of weight 3 and characteristic 5. We develop a method for calculating the decomposition numbers of a Rouquier block based on the work of Chuang and Tan [4].

Finally, in the last chapter, we calculate the decomposition numbers of a Rouquier block, B of $k\mathfrak{S}_{135}$ and its $[3:2]$ pair, $\widetilde{B}$, a block in $k\mathfrak{S}_{133}$. We make use of some theorems in the previous chapter to determine some entries of the decomposition matrix for the latter block.

Chapter 1

Introduction

This thesis is concerned with the modular representation theory of symmetric groups, especially the small defect blocks of symmetric group algebra $k\mathfrak{S}_n$. Our approach is via general representation theory and combinatorial algorithms.

Throughout, k denotes a field of characteristic p. It is well known that a p-block of a symmetric group $\mathfrak{S}_n$ is characterized by its p-core and its weight. The defect group of a block B of $k\mathfrak{S}_n$ is related to the p-weight, denoted by ω, of the block. If the p-weight ω is less than p, then the block has an elementary Abelian defect group of rank ω and hence has defect ω. If $n > \omega p$, then there exist a positive integer i and a block $\widetilde{B}$ of $k\mathfrak{S}_{n-i}$ which forms a $[\omega : i]$-pair with B (Scopes [29]).

Let λ and μ be partitions of n with μ being p-regular. The symmetric group $\mathfrak{S}_n$ has a Specht module S^λ and a p-modular irreducible module D^μ. The decomposition number $[S^\lambda : D^\mu]$ is defined to be the composition multiplicity of D^μ in S^λ. So far, the following facts are known:

1. If $\omega = 0$ or 1 then all the decomposition numbers of the block are 0 or 1.

2. If $\omega = 2$ and $p > 2$ then all the decomposition numbers of the block are 0 or 1 (see [31]).

3. If $\omega = 4$ then some decomposition numbers of a block can be greater than 1, even if $p > w$.

4. If $\omega = 0$, 1, or 2 then there are some methods for determining the decomposition numbers (see Richards [21], Scopes ([29],[30]) and Tan [34].)

The condition for the case $\omega = 3$ is still not properly understood. There is a collection of decomposition numbers for weight 3 which we are unable to evaluate. For example the decomposition number

$$[S^{(2p-2,2p-2,p-1,1)} : D^{(3p-3,2p-1)}]$$

is yet to be determined for $p > 5$.

Using extensive computer calculation, Lubeck and Muller [20] have shown that this decomposition number is equal to 1 for $p = 5$.

Martin and Russel [23] have claimed that all the p-modular decomposition numbers of the symmetric groups of p-weight 3 are 0 or 1 when $p > 3$. Unfortunately, their proof contains a gap when dealing with partitions with p-core $(p-2, p-2)$. This particular case is still an open problem when $p > 5$; as a consequence, the claim in [23] that the decomposition numbers are always 0 or 1 for blocks of partitions of p-weight 3 when $p > 3$ is still open.

In this thesis, we try to determine the decomposition numbers of a Rouquier block B of $k\mathfrak{S}_{135}$ and its $[3:2]$ pair, $\widetilde{B}$, a block in $k\mathfrak{S}_{133}$. We develop a method for calculating the decomposition numbers of a Rouquier block based on the work of Chuang and Tan [4]. We also make use of some theorems listed in [10] to determine some entries of the decomposition matrix for the latter block.

Chapter 2

Preliminaries

In this chapter, we give a summary of the representation theory of finite groups in general and of symmetric groups in particular, which we will need for our work later on. For more elaborate explanation on representation theory of symmetric groups, we refer the readers to [8].

2.1 Group Representation Theory

Block theory

A block of a group algebra kG is an indecomposable two-sided ideal direct summand of kG. A decomposition of kG into blocks

$$kG = B_1 \oplus \ldots \oplus B_s$$

corresponds to a decomposition of the identity element

$$1 = e_1 + \ldots + e_s.$$

Each e_i is called a primitive central idempotent. The correspondence is given by $B_i = e_i kG$.

A kG-module M is said to lie in B_i if e_i is the only primitive central idempotent which does not annihilate M. If M lies in B_i, then M is also a B_i-module. Two kG-module lying in distinct blocks have no common composition factors. Every kG-module can be expressed as a direct sum of modules, each of which lies in a distinct block of kG. So, every indecomposable kG-module lies in a block.

Definition 2.1.1. The *principal block* $B_0(kG)$ of kG is the block in which the trivial module k lies.

The *defect group* D of a block B of kG is a minimal p-subgroup of G, such that all modules lying in B are relatively D-projective. The defect group D of a block is determined up to conjugacy. If $|D| = p^a$ $(a \geq 0)$, then we say that B is a block of defect a.

2.1.1 Restriction and induction

If M is a kG-module and H is a subgroup of G, we get a *restricted* kH-module $M{\downarrow_H}$ by restricting the action of kG to kH. If N is a kH-module, we obtain an *induced* kG-module $N{\uparrow^G}$ by tensoring the regular kG-module with N $(N{\uparrow^G} = kG \otimes_{kH} N)$.

If $b = kGe$ is a block of kG, we write $N{\uparrow^b}$ for $e(N{\uparrow^G})$, the summand(s) of $N{\uparrow^G}$ lying in b. Similarly, if $\widetilde{b} = kH\widetilde{e}$ is a block of kH, we write $M{\downarrow_{\widetilde{b}}}$ for $\widetilde{e}(M{\downarrow_H})$, the summand(s) of $M{\downarrow_H}$ lying in $\widetilde{b}$.

Lemma 2.1.2.

Frobenius Reciprocity:

$$\mathrm{Hom}_{kH}(M{\downarrow_H}, N) \cong \mathrm{Hom}_{kG}(M, N{\uparrow^G}).$$
$$\mathrm{Hom}_{kH}(N, M{\downarrow_H}) \cong \mathrm{Hom}_{kG}(N{\uparrow^G}, M).$$

If K is a subgroup of H and V is a K-module, then

$$(M{\downarrow_H}){\downarrow_K} \cong M{\downarrow_K} \quad \textit{and} \quad (V{\uparrow^H}){\uparrow^G} \cong V{\uparrow^G}.$$

2.1.2 Loewy series and socle series

Definition 2.1.3. The *socle* of a kG module M is the sum of all the irreducible submodules of M, and is written $\mathrm{Soc}(M)$. The socle layers of M are defined inductively by $\mathrm{Soc}^0(M) = 0$, and having defined $\mathrm{Soc}^{n-1}(M)$, we define $\mathrm{Soc}^n(M)$ to be the submodule of M containing $\mathrm{Soc}^{n-1}(M)$ such that $\mathrm{Soc}^n(M)/\mathrm{Soc}^{n-1}(M) = \mathrm{Soc}(M/\mathrm{Soc}^{n-1}(M))$. The *socle series* of M is the filtration

$$0 \subset \mathrm{Soc}(M) \subset \mathrm{Soc}^2(M) \subset \ldots \subset M.$$

We denote the socle structure of M by a matrix whose n-th row, counting from bottom, consists of the composition factors, with multiplicity, of the n-th socle layer of M. The module M is said to be *completely reducible* if $M = \mathrm{Soc}(M)$. This is equivalent to the condition that every submodule has a complement (using Zorn's Lemma) .

Definition 2.1.4. Let G be a finite group. The *radical* J of the group algebra kG is the unique maximal nilpotent (two-sided) ideal of kG. The algebra kG is semi-simple if, and only if, J is trivial. A kG-module M is semi-simple if, and only if, J annihilates M. In particular, $M/J(M)$ is always semi-simple. The *radical series* or *Loewy series* of M is defined inductively by $J^0(M) = M$, $J^n(M) = J(J^{n-1}(M))$. The n-th *radical layer* or *Loewy layer* is $J^{n-1}(M)/J^n(M)$. The *head* of M is $\mathrm{Head}(M) = M/J(M)$, i.e. the first Loewy layer of M. We denote the Loewy structure of M by a matrix whose i-th row consists of precisely the composition factors (with multiplicity) of the i-th Loewy layer of M.

2.2 The Representation Theory of the Symmetric Group

In this section, we give a very short survey on representation of symmetric groups, including the construction of a Specht module. Most of the results are drawn from [7].

2.2.1 Some Basic Definitions

Let k be a field with characteristic p and n be a positive integer.

Definition 2.2.1. The *symmetric group* on n letters, denoted by $\mathfrak{S}_n$, is the group of permutations $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Definition 2.2.2. Given a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers, we say λ is a *partition* of $\{1, 2, \ldots, n\}$, denoted by $\lambda \vdash n$, if $\sum_{i=1}^{\infty} \lambda_i = n$. We also write λ as $(\lambda_1, \ldots, \lambda_m)$ if $\lambda_n = 0$ for $n > m$. A partition is *p-regular* if it does not have p or more equal non-zero parts, otherwise it is called p-singular. The set of partitions of $\{1, 2, \ldots, n\}$ is denoted by $\mathcal{P}(n)$.

Remark. If the sequence is not in weakly decreasing order, we call it an *improper partition*, denoted by $\lambda \models n$.

Definition 2.2.3. Pairs of Partition

Let $\mu = (\mu_1, \mu_2, \ldots)$ be an improper partition of n. A finite sequence of positive integers λ is said to have *type* μ if each i occurs μ_i times in the sequence. We classify the entries of λ inductively, as follows:

1. If $\lambda_i = 1$, then λ_i is of *type I*.

2. If $\lambda_i \geq 2$, and $\mid \{j < i \mid \lambda_j = \lambda_i, \lambda_j \text{ is of type I}\} \mid < \mid \{k < i \mid \lambda_k = \lambda_i - 1, \lambda_k \text{ is of type I}\} \mid$, then λ_i is of *type I*. Otherwise, we say λ_i is of *type II*.

Example 2.2.4. *If $\mu = (3,3,2)$ and $\lambda = (2,2,1,3,1,2,3,1)$, then λ is of type μ and the types of the entries of λ are as follows:*

λ	2	2	1	3	1	2	3	1
type	II	II	I	II	I	I	I	I

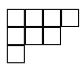
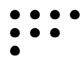
Definition 2.2.5. Given μ an improper partition of n, let $\mu^* = (\mu_1^*, \mu_2^*, \ldots)$ be a sequence of non-negative integers satisfying

$$\mu_{i+1}^* \leq \mu_i^* \leq \mu_i, \text{ for all } i.$$

By definition, it is clear that μ^* is a partition of m, where $m \leq n$. The pair (μ, μ^*) is called a *pair of partitions for n*. The set of sequences of type μ in which, for each i, the number of entries of type I is equals to i is at least μ_i^*, is denoted by $s(\mu^*, \mu)$.

Definition 2.2.6. The *diagram* $[\lambda]$ of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ consists of all ordered pairs (i, j) of integers, called *nodes*, with $1 \leq i \leq m$ and $1 \leq j \leq \lambda_i$. We display the diagram $[\lambda]$ on a two dimensional grid, where each node (i, j) is represented as a square (or dot) on row i and column j. For example,

if $\lambda = (4,3,1)$ then

$[\lambda] =$  (or )

The conjugate partition λ' is the partition whose diagram $[\lambda']$ is the set of nodes $\{(i,j) \mid (j,i) \in [\lambda]\}$, i.e. the diagram of λ' is obtained by interchanging the rows and columns of the diagram of λ.

Example 2.2.7.

$$(5,0,0,0,...) = (5),$$
$$(4,1,0,0,...) = (4,1),$$
$$(2,1,1,1,0,...) = (2,1^3)$$

are partitions of 5 *and the last partition is a* 3*-singular partition. Notice that we have abbreviations such as those written here. The conjugate of the second partition,* $(4,1)$, *is* $(2,1^3)$.

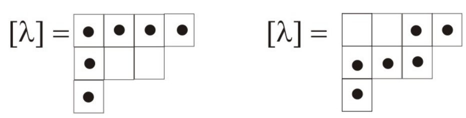
There are two natural orders for $\mathcal{P}(n)$: the *lexicographical order* $\geq$ and the *dominance order* $\trianglerighteq$. If $\lambda, \mu \in \mathcal{P}(n)$, then $\lambda \trianglerighteq \mu$ if and only if $\sum_{i=1}^{r} \lambda_i \geq \sum_{i=1}^{r} \mu_i$ for all $r \in \mathbb{N}$, while $\lambda \geq \mu$ if, and only if, either $\lambda = \mu$ or there exists $l \in \mathbb{N}$ such that $\lambda_l > \mu_l$ and $\lambda_i = \mu_i$ for all $i < l$. So the lexicographical order is a total order extending the dominance order.

Definition 2.2.8. If $\nu = (i,j)$ is a node in the diagram λ, then it has (i,j)-*hook*

$$H_\nu = H_{i,j} = \{(i,k) \mid j \leq k \leq \lambda_i\} \cup \{(k,j) \mid i \leq k \leq \lambda'_j\}$$

where λ'_j is the j-th part of the conjugate partition λ'. The (i,j)-*rim hook* is the connected part of the rim of $[\lambda]$ which begins at the node (i, λ_i) and ends at the node (λ'_j, j). The number of nodes which form a hook is called the *hook length.* A hook of length p is called a p-*hook.* The length of the (i,j)-hook of $[\lambda]$ is denoted by $h^\lambda_{i,j}$. The (i,j)-hook and the (i,j)-rim hook both have length $\lambda_i + \lambda'_j - i - j + 1$. Note that H_ν must be a subset of $[\lambda]$. Both (i,j)-hook and the (i,j)-rim hook have *leg length* equal to $\lambda'_j - i$, i.e. the number of rows below the first row of (i,j)-hook and (i,j)-rim hook respectively.

Example 2.2.9. *If* $\lambda = (4,3,1)$, *then the dotted cells in the left figure is the hook* $H_{1,1}$ *with hook length* $h^\lambda_{1,1} = 6$, *and those in the right figure is the* $(1,1)$*-rim hook.*



Definition 2.2.10. The p-*core* of a partition λ is the unique partition obtained by successively removing hooks of length p from λ and the consecutive partitions

until there is no more hook of length p. The *p-weight* of λ, denoted by ω, is the total number of p-hooks removed to obtain its p-core.

Definition 2.2.11. A node $x \in [\lambda]$ is *removable* if $[\lambda] \setminus \{x\}$ is the diagram of a partition. A node $y \notin [\lambda]$ is *addable* if $[\lambda] \cup \{y\}$ is the diagram of a partition. The node $x = (i, j)$ is called an r-node if r is the residue class of modulo p. A removable r-node $x \in [\lambda]$ is *normal* if whenever y is an addable r-node in $[\lambda]$ which is in a row above x then there are more removable r-nodes between x and y than there are addable r-nodes.

Definition 2.2.12. Suppose $\lambda \vDash n$. A *λ-tableau* is obtained when we replace each node of $\{\lambda\}$ by a positive integer less than or equal to n such that no two of them are equal.

Example 2.2.13.

$$[\lambda] = \begin{array}{llll} 2 & 3 & 4 & 5 \\ 1 & 6 & 7 & \\ 8 & & & \end{array} \quad \textit{and} \quad \begin{array}{llll} 1 & 2 & 3 & 4 \\ 5 & 7 & 8 & \\ 6 & & & \end{array}$$

are $(4, 3, 1)$*-tableaux.*

$\mathfrak{S}_n$ acts naturally (and transitively) on the set of λ-tableaux by permuting the entries of the tableaux.

Let t be a λ-tableaux. We write R_t for the row stabilizer of t in $\mathfrak{S}_n$ i.e. the subgroup of $\mathfrak{S}_n$ which fixes the rows of t set-wise. Similarly, we write C_t for the column stabilizer of t in $\mathfrak{S}_n$.

Example 2.2.14. *If*

$$\begin{array}{llll} 1 & 2 & 3 & 4 \\ 5 & 7 & 8 & \\ 6 & & & \end{array}$$

then

$$R_t = \mathfrak{S}_{\{1,2,3,4\}} \times \mathfrak{S}_{\{5,7,8\}} \times \mathfrak{S}_{\{6\}}$$

$$C_t = \mathfrak{S}_{\{1,5,6\}} \times \mathfrak{S}_{\{2,7\}} \times \mathfrak{S}_{\{3,8\}},$$

where $\mathfrak{S}_X$ is the group of permutations in the set X.

Definition 2.2.15. Given $\lambda \vDash n$, we define an equivalence relation $\sim$ on the set of λ-tableaux as: $t_1 \sim t_2$ if, and only if, $t_1\pi = t_2$ for some $\pi \in R_t$. The equivalence classes are called λ-tabloids, and we write $\{t\}$ for the class containing the λ-tableau t.

It is easy to verify that the action of $\mathfrak{S}_n$ on the set of λ-tableaux, (by permuting the entries in the tableaux), induces an action on the set of λ-tabloids i.e. $\{t\}\pi = \{t\pi\}$ for $\pi \in \mathfrak{S}_n$. As the action of $\mathfrak{S}_n$ on the set λ-tableaux is transitive, so is the action on the set of λ-tabloids.

Definition 2.2.16. Given $\mu \vDash n$, the *Young subgroup* $\mathfrak{S}_\mu$ of $\mathfrak{S}_n$ is

$$\mathfrak{S}_\mu = \mathfrak{S}_{\{1,2,\ldots,\mu_1\}} \times \mathfrak{S}_{\{\mu_1+1,\mu_1+2,\ldots,\mu_1+\mu_2\}} \times \cdots.$$

Let M^μ be the vector space over k whose basis elements are the various μ-tabloids. Extending the action of $\mathfrak{S}_n$ on the set of μ-tabloids linearly to one on M^μ, M^μ becomes an $k\mathfrak{S}_n$-module.

Remark.

1. It is easy to see that M^μ is in fact the permutation module of $k\mathfrak{S}_n$ on $(\mathfrak{S}_n : \mathfrak{S}_\mu)$, the index of $\mathfrak{S}_\mu$ in $\mathfrak{S}_n$. In particular, if $\mu = (1^n)$, then M^μ is the regular representation of $k\mathfrak{S}_n$.

2. Since the action of $\mathfrak{S}_n$ is transitive on the set of μ-tabloids, M^μ is a cyclic $k\mathfrak{S}_n$-module, generated by any μ-tabloid. Also, as $\mathfrak{S}_\mu$ is the stabilizer of one μ-tabloid,
$$\dim M^\mu = \frac{\mathfrak{S}_n}{\mathfrak{S}_\mu} = \frac{n!}{\mu_1!\mu_2!\cdots}$$

If $u \in M^\mu$ and t is a μ-tableau, we say the μ-tabloid $\{t\}$ is *involved* in u if its coefficient in u is non-zero.

Definition 2.2.17. Suppose $\mu \vDash n$ and t is a μ-tableau, we write

$$K_t = \sum_{\pi \in C_t} (\text{sign}\pi)\pi \in k\mathfrak{S}_n.$$

Then the μ-polytabloid e_t associated to t is defined to be $e_t = \{t\}K_t$. The vector subspace S^μ of M^μ spanned by the various μ-polytabloids is called the *Specht module* associated to μ.

Remark.

1. The μ-tabloids involved in e_t have coefficients ± 1.
2. e_t depends on the μ-tableau t rather than its equivalence class $\{t\}$.
3. Since $\pi^{-1}K_t\pi = K_{t\pi}$ for $\pi \in \mathfrak{S}_n$, we have

$$e_t\pi = \{t\}K_t\pi = \{t\}\pi K_{t\pi} = e_{t\pi}$$

So S^μ is a cyclic $k\mathfrak{S}_n$-module, generated by any μ-polytabloid.

2.2.2 Some irreducible representations

In this section, we shall construct some irreducible representation of $\mathfrak{S}_n$ using the Specht modules.

Definition 2.2.18. Suppose $\mu \vDash n$. We define a symmetric, $\mathfrak{S}_n$-invariant, non-degenerated, bilinear form $\langle\ ,\ \rangle$ on M^μ by, if t_1 and t_2 are μ-tableaux,

$$\langle \{t_1\}, \{t_2\} \rangle = \begin{cases} 1, & \text{if} \quad \{t_1\} = \{t_2\}; \\ 0, & \text{otherwise,} \end{cases}$$

and extending it linearly.

If U is a subspace of M^μ, we write $U^\perp$ for the set $\{v \in M^\mu \mid \langle u, v \rangle = 0\ \forall\, \mu \in U\}$.

Remark.

1. $\langle\ ,\ \rangle$ is an inner product when the characteristic of k is 0.

2. If U is $k\mathfrak{S}_n$-submodule of M^μ, then $U^\perp$ is also $k\mathfrak{S}_n$-submodule of M^μ, since $\langle\ ,\ \rangle$ is $\mathfrak{S}_n$-invariant.

We have the following lemmas, the proofs of which can be found in [7].

Lemma 2.2.19. *Suppose $\mu \vdash n$, $u, v \in M^\mu$ and t is a μ-tableau. Then*

$$\langle uK_t, v\rangle = \langle u, vK_t\rangle$$

Lemma 2.2.20. *Suppose $\mu \vdash n$ and $U \subseteq V$ are $k\mathfrak{S}_n$-submodules of M^μ. Then*

$$\frac{U^\perp}{V^\perp} \cong \textit{dual of } \ \frac{V}{U}$$

as $k\mathfrak{S}_n$-modules.

Definition 2.2.21. Suppose $\mu \vdash n$. We define the $k\mathfrak{S}_n$-module D^μ as $D^\mu = S^\mu/S^\mu \cap (S^\mu)^\perp$.

If the characteristic of k is zero, $\langle\ ,\ \rangle$ is an inner-product on M^μ, so that $S^\mu \cup (S^\mu)^\perp = 0$. Hence $S^\mu = D^\mu$. The proof of the next two theorems can be found in [7].

Theorem 2.2.22. *Suppose k has characteristic p. The set $\{D^\mu \mid \mu \vdash n,\ \mu$ p-regular$\}$ is a complete list of mutually non-isomorphic irreducible representations of $\mathfrak{S}_n$ over k. Each D^μ is self-dual and absolutely irreducible.*

Theorem 2.2.23. *If λ is p-regular then S^λ has a unique cosocle D^λ. All the other composition factors of S^λ are of the form D^μ for some $\mu \rhd \lambda$. If λ is p-singular, all the composition factors of S^λ are of the form D^μ for $\mu \rhd \lambda$.*

The multiplicity of D^λ as a composition factor of S^μ is denoted by

$$[S^\mu : D^\lambda]$$

. These numbers are the decomposition numbers $d_{\mu\lambda}$ for the symmetric group $\mathfrak{S}_n$ over the characteristic p.

We now present an amazingly simple product formula for the dimension of Specht modules.

Theorem 2.2.24.

$$\dim S^\lambda = \frac{n!}{\prod_{(i,j)\in[\lambda]} h^\lambda_{i,j}}$$

Proof. See [7]. □

Example 2.2.25. *If $\lambda = (4,3,1)$, then we can draw its hook diagram, where each entry (i,j) is the hook length of $H_{i,j}$, i.e. $h^\lambda_{i,j}$.*

6	4	3	1
4	2	1	
1			

The dimension of $S^{(4,3,1)} = \frac{8!}{6.4.3.4.2} = 70$.

2.2.3 Block representation and the p-abacus

In this section we continue our discussion on block theory and also introduce some new concepts and tools that will be very useful in our subsequent chapters.

Definition 2.2.26. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ is a partition of n. A strictly decreasing sequence $\beta = (\beta_1, \beta_2, \ldots, \beta_u)$ $(u \geq t)$ of non-negative integers is a sequence of β-numbers for λ if

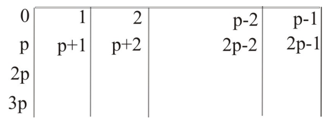
$$\beta_i = \begin{cases} u - i + \lambda_i, & \text{if } 1 \leq i \leq t; \\ u - i, & \text{if } t < i \leq u; \end{cases}$$

The set $\{\beta_1, \ldots, \beta_u\}$ is called a *β-set* for λ. The smallest β-set for λ is the set of first column hook lengths, i.e. $\{h_{i1}^{\lambda} \mid 1 \leq i \leq \lambda_1'\}$.

Conversely, any subset of $\{\psi_i, \ldots, \psi_u\}$ $(\psi_1 > \psi_2 > \ldots > \psi_u)$ of non-negative integers is a β-set for the partition λ defined by $\lambda_i = \psi_i - (u - i)$ $(1 \leq i \leq u)$. There exist more than one β-set for each partition, but every sequence of β-numbers determines a unique partition.

The β-set, and also partition λ, can be visualized using an abacus. There are many advantages of deploying an abacus to represent the partition, for example, the p-cores, p-weights, and also p-quotients (see Definition 2.2.10 and Section 2.2.3) of the partitions can be seen immediately.

An abacus with p-runners is labelled as follows: .



The partition, and its β-set, is displayed on the p-abacus by inserting a bead on position β_i for each element β_i of the β-set. The i-th part of the partition which the β-set defines is the number of unoccupied positions before β_i.

Example 2.2.27. *The β-set $(16, 9, 8, 7, 5, 4, 3, 2, 1, 0)$ corresponding to a partition $(7, 1^3)$ can be displayed using an abacus with three runners (see figure 2.1). However, we will simplify our abacus display by removing its skeleton and we denote the empty space by '$-$'. So our abacus display becomes*

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ - & \bullet & \bullet \\ \bullet & - & - \\ - & - & - \\ - & \bullet & - \end{array}$$

3-abacus display of $(7, 1^3)$ with β-set

$(16, 9, 8, 7, 5, 4, 3, 2, 1, 0)$

If a bead is at position x, then the position $x - 1$ is its preceding position while the position $x + 1$ is its succeeding position.

Given an abacus display for a partition λ, we consider two rearrangements of its beads.

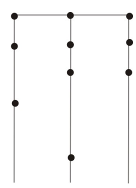


Figure 2.1: Abacus display for $\lambda = (7, 1^3)$

Firstly, moving a bead up one space from position x to an unoccupied position $x-p$ $(x > y)$ corresponds to removing a rim p-hook from the diagram $[\lambda]$.

Secondly, we consider moving a bead from position x to an unoccupied position $x-1$ (resp. $x+1$) corresponds to removing (resp. adding) a node from (resp. to) the initial partition.

In terms of abacus notation, the p-core of λ is the partition whose abacus display is obtained from that of λ by moving all the beads on each runner as far up as they can go. The *p-weight of a bead* is the number of positions it goes up its column to obtain the p-core. The *p-weight of the partition* is the total weight of all the beads.

Theorem 2.2.28. *Each diagram $[\lambda]$ has a uniquely determined p-core.*

Proof. From the paragraphs above, it is clear that the configuration of the p-core is independent of the order in which we move the beads of the initial abacus display of the diagram to get that p-core. □

Next, we introduce the term *p-quotient*. For $i = 1, \ldots, p$, let $\lambda_1^{(i)}$ be the number of unoccupied positions on the i-th runner which occur above the bottommost bead on that runner, $\lambda_2^{(i)}$ be the number of unoccupied positions on the i-th runner which occur above the next-to-bottommost bead on that runner, and so on. Then $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots)$ is a partition, and the p-tuple $(\lambda^{(1)}, \ldots, \lambda^{(p)})$ is called the *p-quotient of* λ. We note that it depends on the number of beads used in the abacus display of λ. The p-weight of λ is equal to $\mid \lambda^{(1)} \mid + \ldots + \mid \lambda^{(p)} \mid$, where $\mid \lambda^i \mid$ denotes the weight of partition $\lambda^{(i)}$

Example 2.2.29. *If $\lambda = (5, 4, 2^2, 1)$, we have the 3 abacus display as follows:*

$$\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & - & \bullet \\
- & \bullet & \bullet \\
- & - & \bullet \\
- & \bullet & -
\end{array}$$

and its 3*-quotient is* $(\emptyset, (2,1), \emptyset)$.

Then, we also have the following:

Theorem 2.2.30. *A diagram $[\lambda]$ is uniquely determined by its p-core and its p-quotient.*

The blocks of $k\mathfrak{S}_n$ can be classified using a theorem better known as Nakayama's 'Conjecture'.

Theorem 2.2.31 ([8] Nakayama's Conjecture). *The Specht modules S^λ and S^μ of $k\mathfrak{S}_n$ lie in the same block if, and only if, λ and μ have the same p-core.*

In terms of the abacus notation, this means that the Specht modules S^λ and S^μ lie in the same block of $k\mathfrak{S}_n$ if and only if for every i, λ and μ can be displayed on abaci with the same number of beads on runner i.

So, if λ and μ are p-regular, and D^λ and D^μ lie in the same block, then λ and μ have the same p-weight, called the *p-weight of the block*. If B is a block of $k\mathfrak{S}_n$ having p-core τ and p-weight ω, then the partitions of B are the elements of $\mathcal{P}(n)$ having p-core τ and p-weight ω.

$S^{(n)}$ is a trivial module. The p-core of (n) is the p-core of the principal block of $k\mathfrak{S}_n$. If $n = p\omega$, then the p-core of $(p\omega)$ is empty, so the principal block of $k\mathfrak{S}_{p\omega}$ has an empty p-core. Every partition of this principal blocks can be represented on an abacus with ω beads on each runner.

Nakayama's Conjecture shows that the number of ordinary irreducible module in the block containing λ is equal to the number of partitions β of n which have the same p-core as λ. A partition λ is uniquely determined by both its p-core and its p-quotient. (See [8].)

Theorem 2.2.32. *[8] The number of ordinary irreducible representations of $k\mathfrak{S}_n$ in a p-block of weight ω is equal to*

$$b(\omega) := \sum p(\omega_1) \cdots p(\omega_p),$$

where $p(m)$ equals the number of partitions of m, and where the sum is taken over all improper partitions $(\omega_1, \cdots, \omega_p)$ of ω.

Theorem 2.2.33. *[8] The number of modular irreducible representations of $k\mathfrak{S}_n$ in a p-block of weight ω is equal to*

$$b^*(\omega) := \sum p(\omega_1) \cdots p(\omega_{p-1}),$$

where $p(m)$ equals the number of partitions of m, and where the sum is taken over all improper partitions $(\omega_1, \cdots, \omega_{p-1})$ of ω. If λ is any partition of n which is of p-weight ω, then $b^*(\omega)$ is equal to the number of p-regular partitions of n with p-cores equal to the p-core of λ.

Lemma 2.2.34. *Let b be a block of $k\mathfrak{S}_n$ with p-weight ω. Then b has defect group $C_p \wr (\mathfrak{S}_\omega)_p$, where $(\mathfrak{S}_\omega)_p$ is a Sylow p-subgroup of $\mathfrak{S}_\omega$. b has defect $\omega + \nu_p(\omega!)$, where ν_p is the standard p-valuation, i.e. if $\nu_p(i) = x$, then x is the largest integer such that p^x divides i.*

2.2.4 Signature representation and Mullineux Algorithm

The symmetric group algebra $k\mathfrak{S}_n$ $(p \geq 3)$ has a natural one-dimensional signature representation $\mathrm{sgn} = \mathrm{sgn}_n$, in which $\mathfrak{S}_n$ acts via $g.v = \mathrm{sgn}(g)v$ $(g \in \mathfrak{S}_n, v \in \mathrm{sgn}_n)$.

Lemma 2.2.35. *[7] Suppose λ is a partition of n. Then $S^\lambda \otimes \mathrm{sgn} \cong$ dual of $S^{\lambda'}$, where λ' is the conjugate partition of λ.*

In [26], Mullineux gave an algorithm which constructs a bijection f from the set of p-regular partitions of n to itself, and conjectured that

$$D^\lambda \otimes \mathrm{sign} = D^{f(\lambda)} \tag{2.1}$$

for all p-regular λ. This conjecture was verified by Ford and Kleshchev [6], by using an equivalent algorithm given in [17]. We shall now describe the algorithm.

For each p-regular partition of λ, we construct a *Mullineux symbol* by removing rim p-hooks from the diagram of λ. Again, we shall describe the process in terms of abacus notation. We form a sequence of partitions $\lambda = \lambda^0, \ldots, \lambda^t = 0$, where λ^i is a partition of some $n_i < n$, and λ^{i+1} is obtained from λ^i by the following algorithm.

1. Let x be the greatest occupied position in the abacus display of λ^i.
2. If there is no unoccupied position less than x in the display, then stop. Otherwise, let y be
   (a) the greatest unoccupied position less than x on the same runner as x, if there is any, or
   (b) the least unoccupied position in the display, if not.

   Move the bead at position x to position y.
3. Let x be the greatest occupied position less than y in the abacus display, and return to step 2.

It is obvious that this procedure will eventually produce the empty partition. Given the partitions $\lambda^0, \ldots, \lambda^t$, define the Mullineux symbol to be the pair of vectors $(r_1, \ldots, r_t)$, $(s_1, \ldots, s_t)$, where

$$r_i = \text{the number of non-zero parts of } \lambda^{i-1},$$
$$s_i = n_{i-1} - n_i.$$

Mullineux showed that a given Mullineux symbol corresponds to at most one p-regular partition, i.e. that a p-regular partition can be reconstructed from its Mullineux symbol. We construct a bijection between Mullineux symbols as follows:

let $((r_1, \ldots, r_t), (s_1, \ldots, s_t))$ correspond to $((r'_1, \ldots, r'_t), (s_1, \ldots, s_t))$, with

$$r'_i = \begin{cases} s_i - r_i, & (p \mid s_i); \\ s_i - r_i + 1, & (p \nmid s_i.) \end{cases}$$

This function is clearly self-inverse. If $((r_1, \ldots, r_t), (s_1, \ldots, s_t))$ corresponds to a p-regular partition of λ of n, then $((r'_1, \ldots, r'_t), (s_1, \ldots, s_t))$ also corresponds to a p-regular partition of n, denoted by $f(\lambda)$. Then Equation 2.1 holds.

2.3 Induction and Restriction of Modules

In our study of representation of the symmetric groups, we make fair use of induction and restriction of modules. In this section we list some results in this area which we shall need.

Restricting a Specht Module to a block of smaller symmetric group corresponds to moving bead(s) on one runner one position to the left (onto the runner left to it). By r-restrict, we mean that we move bead(s) on runner r one position to left (onto runner $r-1$).

Inducing a Specht Module to a block of bigger symmetric group corresponds to moving bead(s) on one runner one position to the right (onto the runner right to it). By r-induce, we mean that we move bead(s) on runner r one position to the right (onto runner $r+1$).

Next, we give a very important theorem.

Theorem 2.3.1. *[7] Branching Rule. Let λ be an element of $\mathcal{P}(n)$. Then*

1. *$S^\lambda\downarrow_{\mathfrak{S}_{n-1}}$ has a Specht filtration (a filtration whose factors are Specht modules). The factors are S^μ with $\mu_i \leq \lambda_i$ for all $i \in \mathbb{N}$. The factor S^μ occurs above S^ν in the filtration if $\mu \rhd \nu$;*

2. *$S^\mu\uparrow^{\mathfrak{S}_n}$ has a Specht filtration. The factors are S^ρ with $\rho_i \leq \mu_i$ for all $i \in \mathbb{N}$. The factor S^ρ occurs above S^σ in the filtration if $\rho \rhd \sigma$.*

Theorem 2.3.2. *[7, 12] (Littlewood-Richardson Rule) In a field of characteristic zero:*

$$(S^\lambda \otimes S^\mu)\uparrow^{\mathfrak{S}_n} \cong \sum c^\nu_{\lambda\mu} S^\nu$$

where $a_\nu = 0$ unless $\lambda_i \leq \nu_i$ for every i, and if $\lambda_i \leq \nu_i$ for every i then a_ν is the number of ways of replacing the nodes of $[\nu] \setminus [\lambda]$ by integers such that the following holds:

1. *The numbers are non-decreasing along rows.*
2. *The numbers are strictly increasing down the columns.*
3. *When reading from right to left in successive rows, we have a sequence in $s(\mu, \mu)$.(See Definition 2.2.3 and Definition 2.2.5.)*

Theorem 2.3.3. *[12] Over an arbitrary field k, $(S^\lambda \otimes S^\mu)\uparrow^{\mathfrak{S}_n}$ has a series with each factor isomorphic to a Specht module for $\mathfrak{S}_n$. The factors occur are those given by the Littlewood-Richardson Rule.*

Next, we introduce some terminologies.

Definition 2.3.4. Let D^λ be a simple module lying in a block of $k\mathfrak{S}_n$. Consider an abacus display for λ. We say a bead b on runner i and in row r of the abacus display is:

1. *normal* if there is no bead directly to the left of b and if for every $j \geq 1$ the number of beads on runner i between rows $r+1$ and $r+j$ (both inclusive) is at least the number of beads on runner $i-1$ between rows $r+1$ and $r+j$ (both inclusive);
2. *good* if b is the highest normal bead on runner i;

3. *conormal* if there is no bead directly to the right of b and if for every $j \geq 1$ the number of beads on runner i between rows $r-1$ and $r-j$ (both inclusive) is at least the number of beads on runner $i+1$ between rows $r-1$ and $r-j$ (both inclusive);

4. *cogood* if b is the lowest conormal bead on runner i.

We now summarize Kleshchev's work [15]-[17] on the restricted simple module $D^\lambda\downarrow_{\widetilde{B}}$. Let $\widehat{B}$ be the block of $k\mathfrak{S}_{n+1}$ whose abacus is obtained by moving a bead from runner i to runner $i+1$, and let $\widetilde{B}$ be the block of $k\mathfrak{S}_{n-1}$ whose abacus is obtained by moving a bead from runner i to runner $i-1$. If b is normal, let λ_b be the partition obtained by moving b one place to the left, and if b is conormal, let λ^b be the partition obtained by moving b one place to the right. With the Definition 2.3.4 and the above definition, we have the following theorem.

Theorem 2.3.5.

1. $D^\lambda\downarrow_{\widetilde{B}} = 0$ *if there is no normal bead on runner* i*. Otherwise* $D^\lambda\downarrow_{\widetilde{B}}$ *is an indecomposable module with simple cosocle and socle both isomorphic to* D^{λ_b}*, where* b *is the unique good bead on runner* i*;* $D^\lambda\downarrow_{\widehat{B}}$ *is simple if, and only if,* b *is the only normal bead on runner* i*.*

2. $D^\lambda\uparrow^{\widehat{B}} = 0$ *if there is no conormal bead on runner* i*. Otherwise* $D^\lambda\uparrow^{\widehat{B}}$ *is an indecomposable module with simple cosocle and socle both isomorphic to* D^{λ^b}*, where* b *is the unique cogood bead on runner* i*;* $D^\lambda\uparrow^{\widehat{B}}$*is simple if, and only if,* b *is the only conormal bead on runner* i*.*

2.3.1 Morita equivalence

Let B be a block of $k\mathfrak{S}_n$ with p-weight ω. Let λ' be the p-core of B and let Γ be a β-set for λ. Let B' be the block whose p-core $\widetilde{\lambda'}$ is obtained by exchanging the two consecutive columns, say the i-th and $(i-1)$-th columns, in the abacus display of the p-core of B (this happens only if the respective columns have a different number of beads on them). This block belongs to $k\mathfrak{S}_{n-d}$ with p-weight ω, where

$$\begin{aligned} d = &\text{number of beads on the } i\text{-th column}- \\ &\text{number of beads on the } (i-1)\text{-th column.} \end{aligned}$$

and has a p-core $\widetilde{\lambda}$.

The two blocks B and B' are said to form a $[\omega : |d|]$-pair.

If $\omega \leq |d|$, then the two blocks are shown to be *Morita equivalent*, and hence have equivalent module categories [30].

Under this equivalence, a simple module D^λ of B corresponds to the simple module $\widetilde{\lambda}$ of B', where $\widetilde{\lambda}$ is the partition obtained by interchanging the $(i-1)$-th and i-th columns of the abacus display of λ. Thus to study the module category of a particular block B of $k\mathfrak{S}_n$, it is enough to look at any block B' with which B forms a $[\omega : |d|]$-pair for some $|d| \geq \omega$. In particular, this shows that all the defect 1 block of symmetric group algebras are Morita equivalent.

Chapter 3

Blocks of Small Defect

We begin with some well known results.

Theorem 3.0.6. *(James & Kerber [8]) The defect group of a symmetric group block having p weight ω is isomorphic to the group C_p wr $(\mathfrak{S}_\omega)_p$ where $(\mathfrak{S}_\omega)_p$ denotes a Sylow subgroup of $\mathfrak{S}_\omega$.*

Corollary 3.0.7. *(James & Kerber [8]) The defect of the block of weight ω is $\omega + \nu_p(\omega!)$ where $\nu_p(\omega!)$ is the highest power of p dividing $\omega!$.*

Lemma 3.0.8. *If $\omega < p$, then the defect group is an abelian group.*

We note that for $\omega < p$, the defect of the block is always the same as its weight.

3.1 Blocks of $k\mathfrak{S}_n$ with Defect 0

Each block of defect 0, thus has weight 0 as well, has only one partition, call it λ and one Specht module, namely S^λ. This Specht module has only one factor, namely D^λ, with multiplicity 1.

3.2 Blocks of $k\mathfrak{S}_n$ with Defect 1

Definition 3.2.1. Let τ be a p-core with r parts and b be an integer not less than $r+p$. Then any p-weight 1 partition with p-core τ may be displayed using an abacus with b beads. The $\langle\ \rangle$-*notation* with b beads is defined as: if the p-weight 1 bead in the abacus display of λ lies on the i-th column, denote λ by $\langle i\rangle$.

Let B be a defect 1 block of $k\mathfrak{S}_n$. This block has a total of p partitions, $p-1$ of which are p-regular. It is easy to see that the partitions of B are totally ordered by the dominance order. Let $\pi \in \mathfrak{S}_p$ such that $i<j$ if, and only if, $\langle\pi(i)\rangle \rhd \langle\pi(j)\rangle$. Then $\langle\pi(1)\rangle \rhd \langle\pi(2)\rangle \rhd \ldots \rhd \langle\pi(p-1)\rangle$ are p-regular, while $\langle\pi(p)\rangle$ is p-singular. The decomposition matrix of B has the following form:

$$\begin{array}{ccccc}
 & D^{\langle\pi(1)\rangle} & D^{\langle\pi(2)\rangle} & \ldots & D^{\langle\pi(p-1)\rangle} \\
S^{\langle\pi(1)\rangle} & 1 & & & \\
S^{\langle\pi(2)\rangle} & 1 & 1 & & \\
\vdots & & \ddots & \ddots & \\
S^{\langle\pi(p-1)\rangle} & & & 1 & 1 \\
S^{\langle\pi(p)\rangle} & & & & 1
\end{array}$$

3.3 Defect 2 Case

We introduce the $\langle\,\rangle$-notation for partitions with p-weight 2.

Definition 3.3.1. Let ν be a p-core with r parts and let b be an integer not less than $r+2p$. Then any p-weight 2 partition with p-core ν may be displayed using an abacus with b beads. The $\langle\ \rangle$-*notation* with b beads is defined as follows: if the abacus display of λ

1. has a bead of p-weight 2 on column i, denote λ by $\langle i \rangle$;

2. has two beads of p-weight 1 on columns i and j, denote λ by $\langle i, j \rangle$.

Theorem 3.3.2. *(Scopes[29] and [31]) Any block of symmetric group with defect 2, over a field k of characteristic $p \geq 3$, has the following properties.*

1. *All decomposition numbers are 0 or 1;*

2. *A row in its decomposition matrix has at most 5 non-zero entries;*

3. *All diagonals entries in its Cartan matrix are greater than or equal to 3;*

4. *All non-diagonal entries in its Cartan matrix are 0,1 or 2;*

5. *The difference between two columns in the decomposition matrix is never non-negative;*

6. *A simple module D^λ extends neither itself nor $D^\lambda \otimes \mathrm{sign}$;*

Her method to prove Theorem 3.3.2 is by induction. The principal block of $k\mathfrak{S}_{2p}$ is the block with the smallest n among all the defect 2 blocks of $k\mathfrak{S}_n$. It is easy to see that the principal block of $k\mathfrak{S}_{2p}$ has all the properties above. This is the base case for our induction. If B is a defect 2 block of $k\mathfrak{S}_n$ with $n > 2p$, then we always can find a column in an abacus display of the p-core of B which has, say d, more beads than the column on its left. Interchanging these columns, we will get the p-core of a defect 2 block B' of $k\mathfrak{S}_{n-d}$. The blocks B and B' form a $[2:d]$-pair. If $d \geq 2$, then the two blocks are Morita equivalent. Since by induction hypothesis, B' has the above properties, so does B. Thus, Scopes only needed to investigate $[2:1]$-pairs. We shall give a summary of what is known about $[2:1]$-pairs.

Let B be a defect 2 block of $k\mathfrak{S}_n$. Suppose the i-th runner in the abacus display of the p-core of B has one more bead than its $(i-1)$-th runner. Let $\widetilde{B}$ be the defect

2 block of $k\mathfrak{S}_{n-1}$ whose p-core is obtained by interchanging the i-th and $(i-1)$-th runners of block B.

Every partition of B has a unique bead on the i-th runner of its abacus display whose preceding position on the $(i-1)$-th runner is empty, except for three partition, the so-called *exceptional partition.* These exceptional partitions have two such beads. We shall name these partitions, $\langle i,i\rangle$, $\langle i,i-1\rangle$ and $\langle i-1\rangle$, as α, β and γ respectively. In the same manner, every partition of $\widetilde{B}$ has a unique bead on the $(i-1)$-th runner of its abacus display whose succeeding position on the i-th runner is empty, except for the three exceptional partitions. We name these exceptional partitions, $\langle i\rangle$, $\langle i-1,i\rangle$ and $\langle i-1,i-1\rangle$, as $\widetilde{\alpha}$, $\widetilde{\beta}$ and $\widetilde{\gamma}$ respectively.

Notation. $M \sim \oplus_{i=1}^{r} M_i$ means that the module M has a filtration

$$M = N_0 \supset N_1 \supset \ldots \supset N_r = 0$$

satisfying

$$\oplus_{i=1}^{r} \frac{N_{i-1}}{N_i} \cong \oplus_{i=1}^{r} M_i.$$

When we restrict and induce the Specht module, corresponding to these partitions of B to $\widetilde{B}$ and $\widetilde{B}$ to B respectively, we get the following relationship:

Lemma 3.3.3. *Let λ be a partition of n and suppose that S^{λ} belongs to the block B. If λ is not equal to α, β or γ, then there exists a unique partition $\widetilde{\lambda}$ of $n-1$, which is not equal to $\widetilde{\alpha}$, $\widetilde{\beta}$ or $\widetilde{\gamma}$, such that $S^{\widetilde{\lambda}}$ belongs to the block $\widetilde{B}$ and*

$$S^{\lambda}{\downarrow}_{\widetilde{B}} \cong S^{\widetilde{\lambda}},$$
$$S^{\widetilde{\lambda}}{\uparrow}^{B} \cong S^{\lambda}.$$

The corresponding results for α, β and γ are

$$\begin{aligned} S^{\alpha}{\downarrow}_{\widetilde{B}} &\sim S^{\widetilde{\alpha}} \oplus S^{\widetilde{\beta}}; & \qquad S^{\widetilde{\alpha}}{\uparrow}^{B} &\sim S^{\alpha} \oplus S^{\beta};\\ S^{\beta}{\downarrow}_{\widetilde{B}} &\sim S^{\widetilde{\alpha}} \oplus S^{\widetilde{\gamma}}; & \qquad S^{\widetilde{\beta}}{\uparrow}^{B} &\sim S^{\alpha} \oplus S^{\gamma};\\ S^{\gamma}{\downarrow}_{\widetilde{B}} &\sim S^{\widetilde{\beta}} \oplus S^{\widetilde{\gamma}}; & \qquad S^{\widetilde{\gamma}}{\uparrow}^{B} &\sim S^{\beta} \oplus S^{\gamma}.\end{aligned}$$

Proof. This follows from the Branching Rule (Theorem 2.3.1). □

Notice that the partitions α and $\widetilde{\alpha}$ are always p-regular, and their corresponding modules D^{α} and $D^{\widetilde{\alpha}}$ occur as composition factors only in S^{α}, S^{β}, S^{γ}, and $S^{\widetilde{\alpha}}$, $S^{\widetilde{\beta}}$, $S^{\widetilde{\gamma}}$ respectively, and not in any other Specht modules.

All simple modules of B are in one-to-one correspondence, related by restriction and induction, with the simple modules of $\widetilde{B}$, except D^{α} and $D^{\widetilde{\alpha}}$.

The restricted simple module $D^{\alpha}{\downarrow}_{\widetilde{B}}$ has two composition factors isomorphic to $D^{\widetilde{\alpha}}$, one as its head and one as its socle. The induced simple module $D^{\alpha}{\uparrow}^{\widetilde{B}}$ has two composition factors isomorphic to D^{α}, one as its head and one as its socle. If β is p-regular (or equivalent to $\widetilde{\gamma}$ being p-regular), then $D^{\beta}{\downarrow}_{\widetilde{B}} \cong D^{\widetilde{\gamma}}$ and $D^{\widetilde{\gamma}}{\uparrow}^{B} \cong D^{\beta}$, and if γ is p-regular (equivalent to $\widetilde{\beta}$ being p-regular), then $D^{\gamma}{\downarrow}_{\widetilde{B}} \cong D^{\widetilde{\beta}}$ and $D^{\widetilde{\beta}}{\uparrow}^{B} \cong D^{\gamma}$.

3.4 Defect 3 Cases

Definition 3.4.1. Let τ be a p-core with r parts and let b be an integer not less than $r + 3p$. Then any p-weight 3 partition with p-core τ may be displayed using an abacus with b beads. The $\langle\ \rangle$- *notation* with b beads is defined as follows: if the abacus display of λ

1. has a bead of p-weight 3 on runner i, denote λ by $\langle i \rangle$;

2. has a bead of p-weight 2 on runner i and a bead of p-weight 1 on runner j, denote λ by $\langle i, j \rangle$;

3. has three beads of p-weight 1 on runner(s) i, j and l, denote λ by $\langle i, j, l \rangle$.

3.4.1 $[3:1]$-pair

We assume that the abacus display of the p-core of a weight 3 block B has one more bead on the i-th runner than on the $(i-1)$-th runner. By interchanging these two runners, we get an abacus display of the p-core of another weight 3 block $\widetilde{B}$, and B and $\widetilde{B}$ form a $[3:1]$-pair.

Definition 3.4.2. With respect to this $[3:1]$-pair:

1. A partition λ of B is called an *exceptional partition* if there are more than one bead on the i-th runner of its abacus display that may be moved to their respective preceding positions on the $(i-1)$-th column. Otherwise, it is called a *non-exceptional partition*.

2. A Specht module S^λ of B is called an *exceptional Specht module* if, and only if, λ is exceptional partition.

3. A simple module D^λ of B is called an *exceptional simple module* if $D^\lambda\downarrow_{\widetilde{B}}$ is not semi-simple. Otherwise, it is called a *non-exceptional simple module*.

4. A partition $\widetilde{\lambda}$ of $\widetilde{B}$ is called an *exceptional partition* if there are more than one bead on the $(i-1)$-th runner of its abacus display that may be moved to their respective succeeding positions on the i-th runner. Otherwise, it is called a *non-exceptional partition*.

5. A Specht module $S^{\widetilde{\lambda}}$ of $\widetilde{B}$ is called an *exceptional Specht module* if, and only if, $\widetilde{\lambda}$ is exceptional.

6. A simple module $D^{\widetilde{\lambda}}$ of $\widetilde{B}$ is *exceptional simple module* if $D^{\widetilde{\lambda}}\uparrow^B$ is not semi-simple. Otherwise, it is a *non-exceptional simple module*.

Remark. As we shall see, the partition associated to an exceptional simple module is always exceptional, but the simple module associated to an exceptional partition (defined when the partition is p-regular) need not be exceptional.

There are $3p$ exceptional partitions of B, denoted as α_i, β_i and γ_i for $(1 \leq i \leq p)$. These partitions have the following abacus displays (we only display the $(i-1)$-th and i-th runner, for $\tau \in \mathfrak{S}_p$) :

$$
\begin{array}{cc}
i{-}1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
- & \bullet \\
- & - \\
- & \bullet
\end{array}
\qquad
\begin{array}{cc}
i{-}1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
\bullet & - \\
- & - \\
- & \bullet
\end{array}
\qquad
\begin{array}{cc}
i{-}1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
- & \bullet \\
- & - \\
\bullet & -
\end{array}
$$

$$\alpha_{\tau(i)} \qquad \beta_{\tau(i)} \qquad \gamma_{\tau(i)}$$

$$
\begin{array}{cc}
i{-}1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
\bullet & \bullet \\
- & \bullet \\
- & \bullet
\end{array}
\qquad
\begin{array}{cc}
i{-}1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
\bullet & \bullet \\
\bullet & - \\
- & \bullet
\end{array}
\qquad
\begin{array}{cc}
i{-}1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
\bullet & \bullet \\
- & \bullet \\
\bullet & -
\end{array}
$$

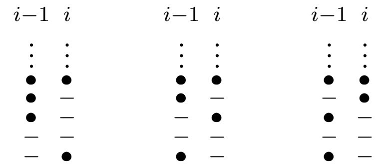
$$\alpha_{\tau(i-1)} \qquad \beta_{\tau(i-1)} \qquad \gamma_{\tau(i-1)}$$

$$
\begin{array}{cc}
i{-}1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
- & \bullet \\
- & \bullet
\end{array}
\qquad
\begin{array}{cc}
i{-}1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
\bullet & - \\
- & \bullet
\end{array}
\qquad
\begin{array}{cc}
i{-}1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
- & \bullet \\
\bullet & -
\end{array}
$$

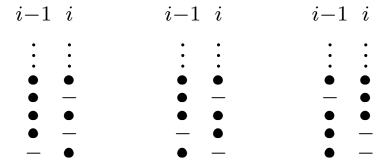
$$\alpha_{\tau(l)} \qquad \beta_{\tau(l)} \qquad \gamma_{\tau(l)}$$

The partitions $\alpha_{\tau(l)}$, $\beta_{\tau(l)}$ and $\gamma_{\tau(l)}$ $(l \neq i, i-1)$ have another bead of p-weight 1 at the runner l. For each m $(1 \leq m \leq p)$, we have $\alpha_{\tau(m)} \rhd \beta_{\tau(m)} \rhd \gamma_{\tau(m)}$. Here, $\tau \in \mathfrak{S}_p$ such that $\alpha_1 > \alpha_2 > \cdots > \alpha_p$. We note that all partitions except α_p are p-regular.

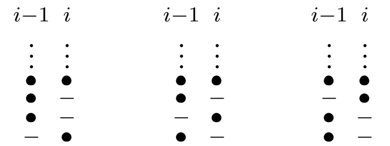
Similarly, there are $3p$ exceptional partitions of $\widetilde{B}$, denoted as $\widetilde{\alpha}_i$, $\widetilde{\beta}_i$ and $\widetilde{\gamma}_i$ for $(1 \leq i \leq p)$. These partitions have the following abacus displays (we only display the $(i-1)$-th and i-th runner, for $\tau \in \mathfrak{S}_p$) :



$\widetilde{\alpha}_{\tau(i)}$ $\widetilde{\beta}_{\tau(i)}$ $\widetilde{\gamma}_{\tau(i)}$



$\widetilde{\alpha}_{\tau(i-1)}$ $\widetilde{\beta}_{\tau(i-1)}$ $\widetilde{\gamma}_{\tau(i-1)}$



$\widetilde{\alpha}_{\tau(l)}$ $\widetilde{\beta}_{\tau(l)}$ $\widetilde{\gamma}_{\tau(l)}$

The partitions $\widetilde{\alpha}_{\tau(l)}$, $\widetilde{\beta}_{\tau(l)}$ and $\widetilde{\gamma}_{\tau(l)}$ $(l \neq i, i-1)$ have another bead of p-weight 1 at runner l. For each m $(1 \leq m \leq p)$, we have $\widetilde{\alpha}_{\tau(m)} \rhd \widetilde{\beta}_{\tau(m)} \rhd \widetilde{\gamma}_{\tau(m)}$. Here, $\tau \in \mathfrak{S}_p$ such that $\alpha_1 > \alpha_2 > \cdots > \alpha_p$. We note that all partitions except $\widetilde{\alpha}_p$ are p-regular.

For every non-exceptional partition, there is a unique bead on runner i on its abacus display that can be moved one position to the left (onto runner $i-1$). Hence, we have the following lemma.

Lemma 3.4.3. *Let λ be a non-exceptional partition and suppose S^λ belongs to block B. Then there is a unique non-exceptional partition $\widetilde{\lambda}$ of $n-1$, such that $\widetilde{\lambda}$ belongs to the block $\widetilde{B}$ and*

$$S^\lambda{\downarrow}_{\widetilde{B}} \sim S^{\widetilde{\lambda}},$$
$$S^{\widetilde{\lambda}}{\uparrow}^{B} \sim S^\lambda.$$

Hence, we have a partial correspondence

$$\Psi : \begin{array}{ccc} B & \to & \widetilde{B} \\ \lambda & \mapsto & \widetilde{\lambda}. \end{array}$$

This map Ψ interchanges runners i and $i-1$ of the associated abacus displays and therefore preserves the lexicographic ordering of partitions and their p-regularities (see [29, 30]. We also have the following relationship as a corollary of the above lemma.

Corollary 3.4.4. *If λ is a p-regular non-exceptional partition, then*

$$D^{\lambda}\downarrow_{\widetilde{B}} \sim D^{\widetilde{\lambda}},$$
$$D^{\widetilde{\lambda}}\uparrow^{B} \sim D^{\lambda}.$$

Meanwhile, the abacus displays of $\alpha_{(m)}$, $\beta_{(m)}$ and $\gamma_{(m)}$ have a unique bead on runner $i-1$ which can be moved one position to the right onto runner i. This action corresponds to inducing the associated Specht modules to a block $\widehat{B}$ of defect 1 of $\mathfrak{S}_{n-1}$. The abacus display of the core of $\widehat{B}$ has one bead less on the runner $i-1$ and one bead more on the runner i than that of the core of B. Hence, we have the following relationship

$$S^{\alpha(m)}\uparrow^{\widehat{B}} \cong S^{\langle\tau(m)\rangle}$$

with $\langle\tau(1)\rangle > \langle\tau(2)\rangle > \cdots > \langle\tau(p)\rangle >$ in $\widehat{B}$. The block $\widehat{B}$ has defect 1. We obtain the following lemma.

Lemma 3.4.5. *[23]*

1. *$D^{\alpha_{\tau(m)}}$ is a composition factor of the Specht modules $S^{\alpha_{\tau(m)}}$, $S^{\beta_{\tau(m)}}$, $S^{\gamma_{\tau(m)}}$, $S^{\alpha_{\tau(m+1)}}$, $S^{\beta_{\tau(m+1)}}$ and $S^{\gamma_{\tau(m+1)}}$, each with multiplicity* 1, *and does not occur in any other Specht module.*

2. *$D^{\beta_{\tau(m)}}$ is a composition factor of $S^{\gamma_{\tau(m)}}$ if $\beta_{\tau(m)}$ is p-regular.*

We obtain a similar result below for the irreducibles $D^{\overline{\alpha}_{\tau(m)}}$, $1 \leq m \leq p-1$, of $\overline{B}$, by considering restriction to a block $\breve{B}$ of defect 1 of $\mathfrak{S}_{n-2}$. The abacus display of the core of $\breve{B}$ has one bead more on runner $i-1$ and one bead less on runner i than that of $\overline{B}$.

Lemma 3.4.6. *[23]*

1. *$D^{\overline{\alpha}_{\tau(m)}}$ is a composition factor of the Specht modules $S^{\overline{\alpha}_{\tau(m)}}$, $S^{\overline{\beta}_{\tau(m)}}$, $S^{\overline{\gamma}_{\tau(m)}}$, $S^{\overline{\alpha}_{\tau(m+1)}}$, $S^{\overline{\beta}_{\tau(m+1)}}$ and $S^{\overline{\gamma}_{\tau(m+1)}}$, each with multiplicity 1, and does not occur in any other Specht module.*

2. *$D^{\overline{\beta}_{\tau(m)}}$ is a composition factor of $S^{\overline{\gamma}_{\tau(m)}}$ if $\overline{\beta}_{\tau(m)}$ is p-regular.*

Martin and Russel claim that the defect 3 blocks have some properties as mentioned below.

Theorem 3.4.7. *(Martin and Russel [22])The block B has the following properties:*

1. *Every entry in its decomposition matrix is either 0 or 1;*

2. *Its simple modules do not self-extend.*

However, James and Mathas [11] have found some gaps in the appendix of [22], and hence the validity of this result is still questionable. The smallest block which [22] has overlooked is that of $k\mathfrak{S}_{5p-4}$ with p-core $(p-2, p-2)$.

Because of space constraint, we do not give much information about $[3:1]$-pairs. For more elaborate treatment on $[3:1]$-pairs, we suggest the reader to refer to [24].

3.4.2 $[3:2]$-pair

In this chapter, we assume that a defect 3 block B of $k\mathfrak{S}_n$ and a defect 3 block $\widetilde{B}$ of $k\mathfrak{S}_{n-2}$ form a $[3:2]$-pair, with the i-th runner of the abacus display of the p-core of B having two beads more than that of $\widetilde{B}$.

We introduce some terminologies.

Definition 3.4.8. Consider $[3:2]$-pair B and $\widetilde{B}$.

1. A partition λ of B is called an *exceptional partition* if we can move more than two beads on the i-th runner of its abacus display to their respective preceding positions on the $(i-1)$-th runner. Otherwise, it is called a *non-exceptional partition.*

2. A Specht module S^λ of B is called an *exceptional Specht module* if, and only if, λ is exceptional partition.

3. A simple module D^λ of B is *exceptional simple module* if $D^\lambda\downarrow_{\widetilde{B}}$ is not semi-simple. Otherwise, it is called a *non-exceptional module.*

4. A partition $\widetilde{\lambda}$ of $\widetilde{B}$ is called an *exceptional partition* if we can move more than two beads of the $(i-1)$-th runner of its abacus display to their respective succeeding positions on the i-th runner. Otherwise, it is called a *non-exceptional partition.*

5. A Specht module $S^{\widetilde{\lambda}}$ of $\widetilde{B}$ is called an *exceptional partition* if, and only if, $\widetilde{\lambda}$ is exceptional partition.

6. A simple module $D^{\widetilde{\lambda}}$ of $\widetilde{B}$ is *exceptional* if $D^{\widetilde{\lambda}}\uparrow^B$ is not semi-simple. Otherwise, it is called a *non-exceptional simple module.*

Definition 3.4.9. For any partition of B, there are beads on the i-th column of its abacus display which may be moved to their respective preceding position on the $(i-1)$-th column. Moving one of these beads corresponds to restricting the associated Specht module of B to a defect 4 block $\overline{B}$. The abacus display of the p-core of $\overline{B}$ has one bead more on the $(i-1)$-th column and one bead less on the i-th column than that of B. Similarly, moving one of the beads on the $(i-1)$-th column in the abacus display of a partition of $\widetilde{B}$ to its succeeding position on the i-th column corresponds to inducing the associated Specht module of $\widetilde{B}$ to $\overline{B}$.

There are four exceptional Specht modules of B whose corresponding partitions have the following $(i-1)$-th and i-th runners in their abacus displays. We denote these Specht modules as S^{α}, S^{β}, S^{γ} and S^{δ}.

$$
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
- & \bullet \\
- & \bullet \\
- & \bullet \\
\multicolumn{2}{c}{\alpha}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
\bullet & - \\
- & \bullet \\
- & \bullet \\
\multicolumn{2}{c}{\beta}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
- & \bullet \\
\bullet & - \\
- & \bullet \\
\multicolumn{2}{c}{\gamma}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
- & \bullet \\
- & \bullet \\
\bullet & - \\
\multicolumn{2}{c}{\delta}
\end{array}
$$

Meanwhile, there are four exceptional Specht modules of $\widetilde{B}$ whose corresponding partitions have the following $(i-1)$-th and i-th runners in their abacus displays. We denote these Specht modules as $S^{\widetilde{\alpha}}$, $S^{\widetilde{\beta}}$, $S^{\widetilde{\gamma}}$ and $S^{\widetilde{\delta}}$.

$$
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
\bullet & - \\
\bullet & - \\
- & \bullet \\
\multicolumn{2}{c}{\widetilde{\alpha}}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
\bullet & - \\
- & \bullet \\
\bullet & - \\
\multicolumn{2}{c}{\widetilde{\beta}}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
- & \bullet \\
\bullet & - \\
\bullet & - \\
\multicolumn{2}{c}{\widetilde{\gamma}}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
\bullet & - \\
\bullet & - \\
\bullet & - \\
\multicolumn{2}{c}{\widetilde{\delta}}
\end{array}
$$

The partitions α, $\widetilde{\alpha}$ and the conjugate partitions δ' and $\widetilde{\delta}'$ are always p-regular. The diagrams below show the dependence of p-regularity among the exceptional partitions and their conjugates.

$$
\begin{array}{ccccccccc}
\beta & p\text{-regular} & \Rightarrow & \gamma & p\text{-regular} & \Rightarrow & \delta & p\text{-regular} \\
 & \Updownarrow & & & \Updownarrow & & & \Updownarrow \\
\widetilde{\delta} & p\text{-regular} & \Rightarrow & \widetilde{\gamma} & p\text{-regular} & \Rightarrow & \widetilde{\beta} & p\text{-regular}
\end{array}
$$

$$\begin{array}{ccccccccc}
\gamma' & p\text{-regular} & \Rightarrow & \beta' & p\text{-regular} & \Rightarrow & \alpha' & p\text{-regular} \\
& \Updownarrow & & & \Updownarrow & & & \Updownarrow \\
\widetilde{\alpha}' & p\text{-regular} & \Rightarrow & \widetilde{\beta}' & p\text{-regular} & \Rightarrow & \widetilde{\gamma}' & p\text{-regular}
\end{array}$$

The restriction of the exceptional Specht modules of B and induction of the exceptional Specht modules of $\widetilde{B}$ are as follows:

$$\begin{array}{ll}
\text{(A1)} & S^{\alpha}{\downarrow_{\widetilde{B}}} \sim 2(S^{\widetilde{\alpha}} \oplus S^{\widetilde{\beta}} \oplus S^{\widetilde{\gamma}}) \\
\text{(A2)} & S^{\beta}{\downarrow_{\widetilde{B}}} \sim 2(S^{\widetilde{\alpha}} \oplus S^{\widetilde{\beta}} \oplus S^{\widetilde{\delta}}) \\
\text{(A3)} & S^{\gamma}{\downarrow_{\widetilde{B}}} \sim 2(S^{\widetilde{\alpha}} \oplus S^{\widetilde{\gamma}} \oplus S^{\widetilde{\delta}}) \\
\text{(A4)} & S^{\delta}{\downarrow_{\widetilde{B}}} \sim 2(S^{\widetilde{\beta}} \oplus S^{\widetilde{\gamma}} \oplus S^{\widetilde{\delta}}) \\
\text{(B1)} & S^{\widetilde{\alpha}}{\uparrow^{B}} \sim 2(S^{\alpha} \oplus S^{\beta} \oplus S^{\gamma}) \\
\text{(B2)} & S^{\widetilde{\beta}}{\uparrow^{B}} \sim 2(S^{\alpha} \oplus S^{\beta} \oplus S^{\delta}) \\
\text{(B3)} & S^{\widetilde{\gamma}}{\uparrow^{B}} \sim 2(S^{\alpha} \oplus S^{\gamma} \oplus S^{\delta}) \\
\text{(B4)} & S^{\widetilde{\delta}}{\uparrow^{B}} \sim 2(S^{\beta} \oplus S^{\gamma} \oplus S^{\delta})
\end{array}$$

For every non-exceptional partition, there are at most two beads on runner i on its abacus display that can be moved one position to the left onto runner $i-1$. Hence, we have the following lemma.

Lemma 3.4.10. *Let λ be a non-exceptional partition of n and suppose that S^{λ} belongs to the block B. Then there is a unique non-exceptional partition $\widetilde{\lambda}$ of $n-2$, such that $S^{\widetilde{\lambda}}$ belongs to the block $\widetilde{B}$. We have the following relationships:*

$$S^{\lambda}{\downarrow_{\widetilde{B}}} \sim 2S^{\widetilde{\lambda}}.$$
$$S^{\widetilde{\lambda}}{\uparrow^{B}} \sim 2S^{\lambda}.$$

Hence, we have a partial correspondence

$$\Psi : \begin{array}{ccc} B & \rightarrow & \widetilde{B} \\ \lambda & \mapsto & \widetilde{\lambda}. \end{array}$$

Remark.

1. As a consequence of the above lemma, the decomposition matrices of B and $\widetilde{B}$ agree on all but four rows which contain S^α, S^β, S^γ, S^δ and $S^{\widetilde{\alpha}}$, $S^{\widetilde{\beta}}$, $S^{\widetilde{\gamma}}$, $S^{\widetilde{\delta}}$ respectively.

2. The map Ψ interchanges runners i and $i-1$ of the associated abacus displays and therefore preserves the lexicographic ordering of partitions and their p-regularity.

There is only one exceptional simple module of B, namely D^α. All other simple modules of B are non-exceptional as they remain semi-simple when restricted to $\widetilde{B}$.

Similarly, there is only one exceptional simple module of $\widetilde{B}$, namely $D^{\widetilde{\alpha}}$. All other simple modules are non-exceptional as they remain semi-simple when induced to B.

We have the following relationship as a corollary of the above lemma.

Corollary 3.4.11. *If λ is a p-regular non-exceptional partition, then*

$$D^\lambda{\downarrow}_{\widetilde{B}} \sim 2D^{\widetilde{\lambda}},$$
$$D^{\widetilde{\lambda}}{\uparrow}^B \sim 2D^\lambda.$$

The restricted module $D^\alpha{\downarrow}_{\widetilde{B}}$ has six copies of $D^{\widetilde{\alpha}}$, and only $D^{\widetilde{\alpha}}$ occurs in its head. Similarly, the induced module $D^{\widetilde{\alpha}}{\uparrow}^B$ has six copies of D^α, and only D^α occurs in its head.

Proposition 3.4.12. *[25, 23]*

1. *The exceptional simple module D^α is a composition factor of S^α, S^β, S^γ and S^δ, each with multiplicity 1. D^α is not a composition factor of any other Specht modules.*

2. *The exceptional simple module $D^{\widetilde{\alpha}}$ is a composition factor of $S^{\widetilde{\alpha}}$, $S^{\widetilde{\beta}}$, $S^{\widetilde{\gamma}}$ and $S^{\widetilde{\delta}}$, each with multiplicity* 1. *$D^{\widetilde{\alpha}}$ is not a composition factor of any other Specht modules.*

Proof. 1. The abacus displays of the exceptional partitions each has a unique bead on runner $i-1$ which can be moved one position to the right onto runner i. This corresponds to inducing the associated Specht module to a block $\hat{B}$ of defect 0 of $\mathfrak{S}_{n+1}$. The abacus display of the single partition, denoted by ν, which belongs to that block, has one bead less on the runner $i-1$ and one bead more on the runner i than that of the core of B. Applying Branching Theorem, we have the following relationship:

$$S^{\alpha}\!\uparrow^{\hat{B}} \cong S^{\nu}, S^{\beta}\!\uparrow^{\hat{B}} \cong S^{\nu}, S^{\gamma}\!\uparrow^{\hat{B}} \cong S^{\nu}, S^{\delta}\!\uparrow^{\hat{B}} \cong S^{\nu}.$$

and $S^{\lambda}\!\uparrow^{\hat{B}} = 0$ for all non-exceptional partitions. Next, we see that S^{α}, S^{β}, S^{γ} and S^{δ} each has precisely one irreducible composition factor which is not a factor of any non-exceptional module, and which gives a copy of S^{ν} on induction to $\mathfrak{S}_{n+1}$. For S^{α}, it must be D^{α}. Schaper's Theorem (See Theorem 4.3.7 in the next chapter) tells us that D^{α} is a composition factor for S^{β}, S^{γ} and S^{δ}, each with multiplicity 1.

2. Consider a restriction to a block $\breve{B}$ of defect 0 of $\mathfrak{S}_{n-2}$. The abacus display of the single partition ν of $\breve{B}$ has one bead more on runner $i-1$ and one bead less on runner i than that of the core of $\widetilde{B}$. The lemma follows using a similar argument as above.

□

We have

Proposition 3.4.13.

1. *If β is p-regular (equivalent to $\widetilde{\delta}$ being p-regular), then $D^{\beta}\downarrow_{\widetilde{B}} \cong 2D^{\widetilde{\delta}}$ and $D^{\widetilde{\delta}}\uparrow^{B} \cong 2D^{\beta}$. Also, D^{β} occurs in both S^{γ} and S^{δ}, each with multiplicity* 1.

2. *If γ is p-regular (equivalent to $\widetilde{\gamma}$ being p-regular), then $D^{\gamma}\downarrow_{\widetilde{B}} \cong 2D^{\widetilde{\gamma}}$ and $D^{\widetilde{\gamma}}\uparrow^{B} \cong 2D^{\gamma}$. Also, D^{γ} and $D^{\widetilde{\gamma}}$ occur with multiplicity 1 in S^{δ} and $S^{\widetilde{\delta}}$ respectively.*

3. *If δ is p-regular (equivalent to $\widetilde{\beta}$ being p-regular), then $D^{\delta}\downarrow_{\widetilde{B}} \cong 2D^{\widetilde{\beta}}$ and $D^{\widetilde{\beta}}\uparrow^{B} \cong 2D^{\delta}$. Also, $D^{\widetilde{\beta}}$ occurs in both $S^{\widetilde{\gamma}}$ and $S^{\widetilde{\delta}}$, each with multiplicity* 1.

Proof. We prove the first part of statements (1), (2) and (3). From (A2), $D^{\beta}\downarrow_{\widetilde{B}}$ has six copies of $D^{\widetilde{\alpha}}$, two copies of $D^{\widetilde{\beta}}$ and two copies of $D^{\widetilde{\delta}}$. Meanwhile, from (A1), $D^{\alpha}\downarrow_{\widetilde{B}}$ has six copies of $D^{\widetilde{\alpha}}$, two copies of $D^{\widetilde{\beta}}$ and two copies of $D^{\widetilde{\gamma}}$. Therefore, $D^{\beta}\downarrow_{\widetilde{B}}$ has no copies of $D^{\widetilde{\alpha}}$ and $D^{\widetilde{\beta}}$ and so $D^{\beta}\downarrow_{\widetilde{B}} \cong 2D^{\widetilde{\delta}}$. The first statement of (2) and (3) follow a similar argument.

The second part of statements (1), (2) and (3) follow from Theorem 4.3.10 and Theorem 4.3.8 in Chapter 4. □

Lemma 3.4.14. *Let M be a B-module. Then $M\downarrow_{\widetilde{B}} \cong (M\downarrow_{\overline{B}})\downarrow_{\widetilde{B}}$. Similarly, if N is a $\widetilde{B}$-module, then $N\uparrow^{B} \cong (N\uparrow^{\overline{B}})\uparrow^{B}$.*

Proof. It is known that $M\downarrow_{\mathfrak{S}_{n-2}} \cong (M\downarrow_{\mathfrak{S}_{n-1}})\downarrow_{\mathfrak{S}_{n-2}}$. Applying the branching rule, we see that the summands of $M\downarrow_{\mathfrak{S}_{n-1}}$ which do not lie in $\overline{B}$ vanish when they are restricted to $\widetilde{B}$. So, we can conclude that $(M\downarrow_{\mathfrak{S}_{n-1}})\downarrow_{\widetilde{B}} \cong (M\downarrow_{\overline{B}})\downarrow_{\widetilde{B}}$. The first statement now follows. The second statement can be verified using the same argument. □

Every Specht module of $\overline{B}$ restricts to a unique Specht module of $\widetilde{B}$ (or gives zero) and induces to a unique Specht module of B (or gives zero), with six exceptions. These exceptional Specht modules will be denoted as $S^{\overline{\alpha}}$, $S^{\overline{\beta}}$, $S^{\overline{\gamma}}$, $S^{\overline{\delta}}$, $S^{\overline{\epsilon}}$ and

$S^{\overline{\kappa}}$. Their corresponding partitions have the following $(i-1)$-th and i-th runners in the abacus display

$$
\begin{array}{cc} i-1 & i \\ \vdots & \vdots \\ \bullet & \bullet \\ \bullet & - \\ \bullet & - \\ & \bullet \\ - & \bullet \\ \multicolumn{2}{c}{\overline{\alpha}} \end{array}
\qquad
\begin{array}{cc} i-1 & i \\ \vdots & \vdots \\ \bullet & \bullet \\ \bullet & - \\ - & \bullet \\ \bullet & - \\ - & \bullet \\ \multicolumn{2}{c}{\overline{\beta}} \end{array}
\qquad
\begin{array}{cc} i-1 & i \\ \vdots & \vdots \\ \bullet & \bullet \\ - & \bullet \\ \bullet & - \\ \bullet & - \\ - & \bullet \\ \multicolumn{2}{c}{\overline{\gamma}} \end{array}
\qquad
\begin{array}{cc} i-1 & i \\ \vdots & \vdots \\ \bullet & \bullet \\ \bullet & - \\ - & \bullet \\ - & \bullet \\ \bullet & - \\ \multicolumn{2}{c}{\overline{\delta}} \end{array}
\qquad
\begin{array}{cc} i-1 & i \\ \vdots & \vdots \\ \bullet & \bullet \\ - & \bullet \\ \bullet & - \\ - & \bullet \\ \bullet & - \\ \multicolumn{2}{c}{\overline{\epsilon}} \end{array}
\qquad
\begin{array}{cc} i-1 & i \\ \vdots & \vdots \\ \bullet & \bullet \\ - & \bullet \\ - & \bullet \\ \bullet & - \\ \bullet & - \\ \multicolumn{2}{c}{\overline{\kappa}} \end{array}
$$

These Specht modules of $\overline{B}$ have the following relationships, by Theorem 2.3.1, with the exceptional Specht modules of $\widetilde{B}$ and B:

$$
\begin{array}{lcc}
\text{(C1)} & S^{\overline{\alpha}}\!\downarrow_{\widetilde{B}} \sim S^{\widetilde{\alpha}} \oplus S^{\widetilde{\beta}}; & S^{\overline{\alpha}}\!\uparrow^{B} \sim S^{\alpha} \oplus S^{\beta}; \\
\text{(C2)} & S^{\overline{\beta}}\!\downarrow_{\widetilde{B}} \sim S^{\widetilde{\alpha}} \oplus S^{\widetilde{\gamma}}; & S^{\overline{\beta}}\!\uparrow^{B} \sim S^{\alpha} \oplus S^{\gamma}; \\
\text{(C3)} & S^{\overline{\gamma}}\!\downarrow_{\widetilde{B}} \sim S^{\widetilde{\alpha}} \oplus S^{\widetilde{\delta}}; & S^{\overline{\gamma}}\!\uparrow^{B} \sim S^{\beta} \oplus S^{\gamma}; \\
\text{(C4)} & S^{\overline{\delta}}\!\downarrow_{\widetilde{B}} \sim S^{\widetilde{\beta}} \oplus S^{\widetilde{\gamma}}; & S^{\overline{\delta}}\!\uparrow^{B} \sim S^{\alpha} \oplus S^{\delta}; \\
\text{(C5)} & S^{\overline{\epsilon}}\!\downarrow_{\widetilde{B}} \sim S^{\widetilde{\beta}} \oplus S^{\widetilde{\delta}}; & S^{\overline{\epsilon}}\!\uparrow^{B} \sim S^{\beta} \oplus S^{\delta}; \\
\text{(C6)} & S^{\overline{\kappa}}\!\downarrow_{\widetilde{B}} \sim S^{\widetilde{\gamma}} \oplus S^{\widetilde{\delta}}; & S^{\overline{\kappa}}\!\uparrow^{B} \sim S^{\gamma} \oplus S^{\delta}; \\
\text{(D1)} & S^{\widetilde{\alpha}}\!\uparrow^{\overline{B}} \sim S^{\overline{\alpha}} \oplus S^{\overline{\beta}} \oplus S^{\overline{\gamma}}; & S^{\alpha}\!\downarrow_{\overline{B}} \sim S^{\overline{\alpha}} \oplus S^{\overline{\beta}} \oplus S^{\overline{\delta}}; \\
\text{(D2)} & S^{\widetilde{\beta}}\!\uparrow^{\overline{B}} \sim S^{\overline{\alpha}} \oplus S^{\overline{\delta}} \oplus S^{\overline{\epsilon}}; & S^{\beta}\!\downarrow_{\overline{B}} \sim S^{\overline{\alpha}} \oplus S^{\overline{\gamma}} \oplus S^{\overline{\epsilon}}; \\
\text{(D3)} & S^{\widetilde{\gamma}}\!\uparrow^{\overline{B}} \sim S^{\overline{\beta}} \oplus S^{\overline{\delta}} \oplus S^{\overline{\kappa}}; & S^{\gamma}\!\downarrow_{\overline{B}} \sim S^{\overline{\beta}} \oplus S^{\overline{\gamma}} \oplus S^{\overline{\kappa}}; \\
\text{(D4)} & S^{\widetilde{\delta}}\!\uparrow^{\overline{B}} \sim S^{\overline{\gamma}} \oplus S^{\overline{\epsilon}} \oplus S^{\overline{\kappa}}; & S^{\delta}\!\downarrow_{\overline{B}} \sim S^{\overline{\delta}} \oplus S^{\overline{\epsilon}} \oplus S^{\overline{\kappa}}.
\end{array}
$$

For each non-exceptional Specht module $S^{\widetilde{\lambda}}$ of $\overline{B}$ with $S^{\widetilde{\lambda}}\!\uparrow^{B} \sim 2S^{\lambda}$, there exist two Specht modules $S^{\overline{\lambda}}$ and $S^{\overline{\mu}}$ of $\overline{B}$ (with $\overline{\lambda} > \overline{\mu}$) such that $S^{\widetilde{\lambda}}\!\uparrow^{\overline{B}} \sim S^{\overline{\lambda}} \oplus S^{\overline{\mu}} \sim S^{\lambda}\!\downarrow_{\overline{B}}$, $S^{\overline{\lambda}}\!\downarrow_{\widetilde{B}} \cong S^{\overline{\mu}}\!\downarrow_{\widetilde{B}} \cong S^{\widetilde{\lambda}}$ and $S^{\overline{\lambda}}\!\uparrow^{B} \cong S^{\overline{\mu}}\!\uparrow^{B} \cong S^{\lambda}$.

Lemma 3.4.15. *Suppose $D^{\widetilde{\lambda}}$ is a non-exceptional simple module of $\widetilde{B}$. Then $(D^{\widetilde{\lambda}}\!\uparrow^{\overline{B}})\!\downarrow_{\widetilde{B}} \cong 2D^{\widetilde{\lambda}}$. Similarly, if D^{λ} is a non-exceptional simple module of B, then $(D^{\lambda}\!\downarrow_{\overline{B}})\!\uparrow^{B} \cong 2D^{\lambda}$.*

Proposition 3.4.16. *[25] For each non-exceptional simple module D^{λ} of B with $D^{\lambda}\!\downarrow_{\widetilde{B}} \cong 2D^{\widetilde{\lambda}}$, there exists a unique simple module $D^{\overline{\lambda}}$ of $\overline{B}$ such that*

1. *$D^{\overline{\lambda}}\uparrow^B \cong D^{\lambda}$;*

2. *$D^{\lambda}\downarrow_{\overline{B}}$ is non-simple, has head and socle both isomorphic to $D^{\overline{\lambda}}$, and all the composition factors $D^{\overline{\mu}}$ in its heart satisfy $D^{\overline{\mu}}\uparrow^B = 0$;*

3. *$D^{\overline{\lambda}}\downarrow_{\widetilde{B}} \cong D^{\widetilde{\lambda}}$;*

4. *$D^{\widetilde{\lambda}}\uparrow^{\overline{B}}$ is non-simple, has head and socle both isomorphic to $D^{\overline{\lambda}}$, and all the composition factors $D^{\overline{\mu}}$ in its heart satisfy $D^{\overline{\mu}}\downarrow_{\widetilde{B}} = 0$;*

5. *$D^{\widetilde{\lambda}}\uparrow^{\overline{B}} \cong D^{\lambda}\downarrow_{\overline{B}}$.*

Lemma 3.4.17.

$$\begin{aligned} &[D^{\alpha}\downarrow_{\overline{B}} : D^{\overline{\alpha}}] = 3; && [D^{\widetilde{\alpha}}\uparrow^{\overline{B}} : D^{\overline{\alpha}}] = 3; \\ &[D^{\overline{\alpha}}\uparrow^{B} : D^{\alpha}] = 2; && [D^{\overline{\alpha}}\downarrow_{\widetilde{B}} : D^{\widetilde{\alpha}}] = 2; \end{aligned}$$

$D^{\overline{\lambda}}$ is a composition factor of $S^{\overline{\alpha}}$, $S^{\overline{\beta}}$, $S^{\overline{\gamma}}$, $S^{\overline{\delta}}$, $S^{\overline{\epsilon}}$ and $S^{\overline{\kappa}}$ and each has multiplicity 1. *$D^{\overline{\lambda}}$ is not a composition of any other Specht module.*

Proof. By using Proposition 3.4.16 and relationship (C1), it is easy to verify that the multiplicity of $[D^{\overline{\alpha}}\downarrow_{\widetilde{B}} : D^{\widetilde{\alpha}}]$ and the multiplicity of $[D^{\overline{\alpha}}\uparrow^{B} : D^{\alpha}]$ are equal to 2. From relationship (C2)-(C6), we know that $D^{\overline{\alpha}}$ occurs at most once in each of the exceptional Specht module of $\overline{B}$. We give one example and the rest follow the same argument. $[S^{\overline{\gamma}}\downarrow_{\widetilde{B}} : D^{\widetilde{\alpha}}] = 2$ by relationship (C3), and since $[D^{\overline{\alpha}}\downarrow_{\widetilde{B}} : D^{\widetilde{\alpha}}] = 2$, we see that $D^{\overline{\alpha}}$ can at most occur once in $S^{\overline{\gamma}}$. Kleshchev's result on restricted simple modules [17] shows that $[D^{\alpha}\downarrow_{\overline{B}} : D^{\overline{\alpha}}] = 3$. Then, together with relationships (D1)-(D4), we see that $D^{\overline{\lambda}}$ must occur exactly once in each of exceptional Specht modules of $\overline{B}$. Relationship (D2) shows that $[S^{\overline{\alpha}} \oplus S^{\overline{\gamma}} \oplus S^{\overline{\epsilon}} : D^{\overline{\alpha}}] \geq 3$; and this leads $D^{\overline{\alpha}}$ to occur exactly once in each of $S^{\overline{\alpha}}$,$S^{\overline{\gamma}}$ and $S^{\overline{\epsilon}}$. Lastly, the multiplicity of $[D^{\widetilde{\alpha}}\uparrow^{\overline{B}} : D^{\overline{\alpha}}]$ can be verified using (D1). If $D^{\overline{\alpha}}$ is a composition factor of $S^{\overline{\lambda}}$, then $D^{\widetilde{\alpha}}$ is a composition factor of $S^{\overline{\lambda}}\downarrow_{\widetilde{B}}$. But we know that $S^{\overline{\lambda}}$ is not exceptional, so $S^{\overline{\lambda}}{\downarrow}_{\widetilde{B}} \cong S^{\widetilde{\alpha}}$ is also not exceptional, thus will not have $D^{\widetilde{\alpha}}$ as a composition factor. $\square$

For more elaborate treatment on [3 : 2]-pairs, we suggest the reader to refer to [25] and [23].

Chapter 4

Calculating the Decomposition Numbers

4.1 Blocks of Weight 0, 1 and 2

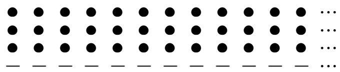
As mentioned in chapter one, for blocks of weight 0 and 1, it is well known that all the decomposition numbers are either 0 or 1. Particulary, in section (3.2), we already show a general form of decomposition matrix for blocks of weight 1. Also for $\omega = 2$, there is a known method for determining the decomposition numbers [21].

4.2 Blocks of Weight 3

For this section, we fix the p-weight $\omega = 3$ and assume that $p > 3$. James and Mathas [10] found that by iterating the process of employing Corollary 4.3.4, they can reduce the calculation of *all* decomposition numbers for blocks of weight 3 to considering only certain blocks, and equivalently, certain p-cores, by Nakayama's 'Conjecture' [10]. We describe the abaci for this minimal collection of p-cores. Without lost of generality, we may assume that each of our abaci has exactly 3 beads on runner 1 and at least 3 beads on every other runner. There are 4 cases to be considered [10].

Case 1.

All of the runners contain exactly 3 beads.



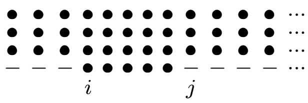
In this case, the p-core is empty. For this case, we can make use of Corollary 4.3.5 and the remark following it.

Remark. Case 1 is the only case to be considered if $\omega = 1$.

Case 2.

In this case, the abacus display of the block has the following configuration:

Each of the first $i-1$ runners contains exactly 3 beads, runners i up to $j-1$ each contains 4 beads and runners j to p each contains 3 beads, where $1 < i < j \leq p+1$. There are $\binom{p}{2}$ such p-cores.



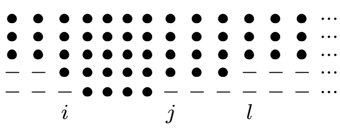
The p-core is $(i-1)^{j-i}$, a partition of $(ij - i^2 + i - j)$.

Remark. Case 1 and 2 are the only cases to be considered if $\omega = 2$.

Case 3.

In this case, the abacus display of the block has the following configuration:

Each of the first $i-1$ runners contains exactly 3 beads, runner i contains 4 beads, runners $i+1$ up to $j-1$ each contains 5 beads, runners j to $l-1$ each contains 4 beads and runner l to p each contains 3 beads, where $2 < i+1 < j \leq l \leq p+1$. There are $\binom{p-1}{3} + \binom{p-1}{2} = \binom{p}{3}$ such p-cores.



The p-core is $((p-l+2i)^{j-i-1}, (i-1)^{l-i})$.

Case 4.

In this case, the abacus display of the block has configuration as follows: The first $i-1$ runners contain exactly 3 beads; runners i to $j-1$ contain 5 beads; and runners j to $l-1$ contain 4 beads; runners l to p contain 3 beads, where $1 < i < j \leq l \leq p+1$. There are $\binom{p}{3} + \binom{p}{2} = \binom{p+1}{3}$ p-cores.

$$
\begin{array}{cccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots \\
- & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - & \cdots \\
- & - & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - & - & \cdots \\
 & & i & & & & & j & & & l & & &
\end{array}
$$

The p core is $((p-l+2i-1)^{j-i}, (i-1)^{l-i})$.

4.3 Method for Calculating the Decomposition Number

4.3.1 Some Basic Results

Scopes and James [29, 30, 23] did some pioneering works of some basic techniques for calculating the decomposition numbers. By using the results of Brundan and Kleshchev [14], James and Mathas [10] improve the presentation of these methods, which we will explore in this chapter.

We start with a simple proposition.

Proposition 4.3.1. *Assume that λ and μ are partitions of n with μ being p-regular, and that m is a positive integer such that*

1. *λ has at most m removable r-nodes,*

2. *μ has at least m normal r-nodes.*

Let l be the number of removable r-nodes of λ. Then

1. *If $l < m$ then $[S^\lambda : D^\mu] = 0$.*

2. *If $l = m$, then $[S^\lambda : D^\mu] = [S^{\overline{\lambda}} : D^{\overline{\mu}}]$, where $\overline{\lambda}$ is the partition obtained from λ by removing its m r-nodes, and $\overline{\mu}$ is the partition obtained from μ by removing its lowest m normal r-nodes.*

Proof. We r-restrict D^μ m-times by moving the m normal beads on runner r of the abacus display of μ. We then obtain a $k\mathfrak{S}_{n-m}$-module which contains $D^{\overline{\mu}}$ as a submodule. However if λ has fewer than m removable r-nodes then when we r-restrict S^λ m times, we will obtain the zero module, thus $[S^\lambda : D^\mu] = 0$. Part (2) of the proposition is obtained using Lemma 2.13 of [14]. $\square$

From now on we will use the abacus display extensively. We modify the definition of r-node introduced in Definition 2.2.11.

Definition 4.3.2. A bead in the abacus display is called an r-*node* if it lies on runner r.

This changes the previous definition of r-node (Definition 2.2.11) by a constant and it is harmless. With this convention, removing an r-node from a partition corresponds to moving a bead on runner r to runner $r-1$ at the same level if $r \neq 1$ and one level higher if $r = 1$ (i.e. from runner 1 to runner p). Adding an r-node corresponds to moving a bead on runner $r-1$ to runner r at the same level if $r \neq p$ and one level lower if $r = p$ (i.e. from runner p to runner 1).

Using Nakayama's 'Conjecture' and its consequence as mentioned in the previous chapter, we will now present some corollaries of Proposition 4.3.1.

Corollary 4.3.3. *Suppose that the partition λ of n has exactly m removable r-nodes and no addable r-nodes. Let μ be a p-regular partition of n in the same block as λ. Then $[S^\lambda : D^\mu]$ is equal to an explicit decomposition number of $\mathfrak{S}_{n-k}$ which is in a block of the same weight as λ.*

Proof. μ has exactly m more removable r-nodes than addable r-nodes and so has at least m normal r-nodes. The corollary now follows immediately from Proposition 4.3.1. Note that the block of $\mathfrak{S}_{n-k}$ has the same weight as λ because the abacus display of $\overline{\lambda}$ can be obtained by interchanging runners $r-1$ and r of abacus display of λ. □

Corollary 4.3.4. *Suppose that B is a block of $k\mathfrak{S}_n$ with the property that for every partition in B, there exists an r such that the partition has a removable r-node but no addable r-node. Then we can equate each decomposition number of B with an explicit decomposition number for a smaller symmetric group.*

Corollary 4.3.5. *Suppose that $\omega \leq 3$. Then every decomposition number for the principal block of $k\mathfrak{S}_{\omega p}$ is either zero or can be equated with an explicit decomposition number of $k\mathfrak{S}_{\omega p-1}$.*

Proof. Let S^λ be a Specht module belongs to the principal block of $\mathfrak{S}_{\omega p}$. We look at the abacus display of λ. As $\omega \leq 3$, for each r, we can move at most one bead from runner r to runner $r-1$, i.e. λ has at most one removable r-node. Suppose μ is p-regular. Choose runner r from abacus display of μ such that it has one normal r-node. Now we can apply Proposition 4.3.1 to obtain the desired result. □

Remark. The weight of a partition of $k\mathfrak{S}_{\omega p-1}$ must be less than ω, and all the decomposition numbers for blocks of weight 0, 1 and 2 are known. Therefore, Corollary 4.3.5 determines the decomposition numbers of $k\mathfrak{S}_{3p}$. The proof above fails when $\omega = 4$ because λ may have more than one removable r-nodes. We give a counterexample. Let $p = 3$ and consider the partition $\lambda = (6, 4, 1^2)$ which has the abacus display:

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & - \\ \bullet & \bullet & - \\ - & - & \bullet \\ - & - & \bullet \end{array}$$

λ has p-weight equal to 4, and from the abacus display, it is easy to see that λ has two removable r-nodes.

4.3.2 Some methods for calculating decomposition numbers

We present a collection of techniques for calculating or at least estimating the decomposition numbers. Most of the materials in this section are drawn from [10] with some modifications.

With these methods, we can determine completely the decomposition numbers for blocks of weight 0, 1 or 2, and deal with blocks of higher weight to some extent.

Keep in mind that we may assume that λ and μ are in the same block and that $\mu \trianglerighteq \lambda$, since otherwise $[S^\lambda : D^\mu] = 0$. In particular, the number of non-zero parts of μ cannot exceed the number of non zero parts of λ. Hence, whatever abacus we use to represent λ can also be used to represent μ. This is true because the number of parts of a partition can be read off from its abacus display by counting the number of beads after the first gap.

Recall the definition of normal nodes in the abacus display, (Definition 2.3.4). Our first method is actually the abacus version of Corollary 4.3.3.

Theorem 4.3.6. *[10] Suppose that λ has an abacus display such that for some r, there are exactly m beads on runner r that can be moved one position to the left and that there are no beads on runner $r-1$ that can be moved one position to the right. Then $[S^\lambda : D^\mu] = [S^{\overline{\lambda}} : D^{\overline{\mu}}]$, where the abacus display for $\overline{\lambda}$ is obtained from that for λ by moving the m possible beads on runner r one position to the left, and the abacus display for $\overline{\mu}$ is obtained by moving the m beads on runner r corresponding to the lowest m normal nodes in μ one position to the left.*

Remark.

1. Usually μ has no addable r-nodes, so to get $\overline{\mu}$, one only need to move the m possible beads on runner r one position to the left.

2. If $m \geq 1$, then Theorem 4.3.6 in fact says that $[S^\lambda : D^\mu]$ in $k\mathfrak{S}_n$ is equal to a decomposition number in a block in $k\mathfrak{S}_m$, $m < n$, which has the same weight as λ.

The next method is developed by Schaper [28].

Theorem 4.3.7. *(Schaper's Theorem). Suppose λ and μ are distinct partitions of a block B of $k\mathfrak{S}_n$ with μ being p-regular. Denote the entry $[S^\rho : D^\sigma]$ in the decomposition matrix of B by $d_{\rho\sigma}$. Define*

$$b_{\lambda\mu} = \sum_{i=1}^{p-weight\ of\ B} (\nu_p(i)+1) \sum_\tau \sum_\sigma ((-1)^{\tau\sigma} d_{\sigma\mu}) \tag{4.1}$$

where the second and third sums are respectively over partitions $\tau \in \mathcal{P}(n-ip)$ obtained by moving a bead in the abacus display of λ, i positions up its runner and over partitions $\sigma \in \mathcal{P}$ with $\sigma > \lambda$ obtained by moving a bead in the abacus display of τ, i positions down its runner, and where

$$(-1)^{\tau\sigma} = \begin{cases} 1, & \textit{if the number of beads crossed by the bead moved to} \\ & \textit{obtain } \tau \textit{ is of different parity from that to obtain } \sigma; \\ -1, & \textit{otherwise.} \end{cases}$$

(If $\nu_p(i) = x$, then x is the largest integer such that p^x divides i.) Then $b_{\lambda\mu} \geq d_{\lambda\mu}$, and $b_{\lambda\mu} = 0$ if, and only if, $d_{\lambda\mu} = 0$.

We will show how this theorem works later on.

We move on to present some methods developed by G.D. James. The next theorem says that if the diagrams of λ and μ have the same first columns, then we can remove their first columns and get the new partitions $\overline{\lambda}$ and $\overline{\mu}$. Their decomposition number is exactly the same as $[S^\lambda : D^\mu]$.

Theorem 4.3.8. *Assume that $\lambda = \{\lambda_1, \dots, \lambda_k\}$ and $\mu = \{\mu_1, \dots, \mu_k\}$ are partitions of n, with μ being p-regular, and suppose that λ has exactly k non-zero parts. If $\mu_{k+1} = 0$ and $\mu_k > 0$, i.e. λ and μ have the same first columns, then*

$$[S^\lambda : D^\mu] = [S^{(\lambda_1-1,\lambda_2-1,\dots,\lambda_k-1)} : D^{(\mu_1-1,\mu_2-1,\dots,\mu_k-1)}]. \tag{4.2}$$

Proof. See [9], Theorem 6. □

Remark.

1. If $\mu \ntrianglerighteq \lambda$ then $[S^\lambda : D^\mu] = 0$. See Theorem 2.2.23.
2. If $\mu_{k+1} > 0$ then $[S^\lambda : D^\mu] = 0$. The reason is as follows. If $\mu_{k+1} > 0$ then $\mu \ntrianglerighteq \lambda$ so $[S^\lambda : D^\mu] = 0$.
3. Removing the first column from a partition is equivalent to putting a bead in the first gap of the abacus display.

The next theorem in fact is a generalization of the theorem above.

Theorem 4.3.9. *Suppose $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_m)$. Suppose there exists an r such that*

$$\lambda_1 + \dots + \lambda_r = \mu_1 + \dots + \mu_r,$$
$$\lambda_{r+1} + \dots + \lambda_n = \mu_{r+1} + \dots + \mu_m.$$

Let

$$\lambda_{(1)} = (\lambda_1, \dots, \lambda_r),$$
$$\lambda_{(2)} = (\lambda_{r+1}, \dots, \lambda_n),$$
$$\mu_{(1)} = (\mu_1, \dots, \mu_r),$$
$$\mu_{(2)} = (\mu_{r+1}, \dots, \mu_m).$$

Then

$$[S^\lambda : D^\mu] = [S^{\lambda_{(1)}} : D^{\mu_{(1)}}][S^{\lambda_{(2)}} : D^{\mu_{(2)}}].$$

Proof. See [5]. □

The theorem below says that if the first row of the diagram of λ is equal to the first row of the diagram of μ, then we can remove these first rows without changing the decomposition number $[S^\lambda : D^\mu]$.

Theorem 4.3.10. *[9] Assume that $\lambda = \{\lambda_1, \dots, \lambda_k\}$ and $\mu = \{\mu_1, \dots, \mu_l\}$ are partitions of n with μ being p-regular, and $\lambda_1 = \mu_1$. Let*

$$\overline{\lambda} = (\lambda_2, \lambda_3, \dots, \lambda_k), \quad \textit{and} \quad \overline{\mu} = (\mu_2, \mu_3, \dots, \mu_l).$$

Then $[S^\lambda : D^\mu] = [S^{\overline{\lambda}} : D^{\overline{\mu}}]$.

Remark. Removing the first row from a partition is equivalent to removing the last bead in the abacus display.

We have the rule similar with Theorem 4.3.9.

Theorem 4.3.11. *Suppose $\lambda' = (c_1, \dots, c_s)$ and $\mu' = (d_1, \dots, d_t)$ are the conjugate partitions of λ and μ respectively. Suppose there exists an r such that*

$$c_1 + \ldots + c_r = d_1 + \ldots + d_r,$$
$$c_{r+1} + \ldots + c_s = d_{r+1} + \ldots + d_t.$$

Let

$$\lambda'_{(1)} = (c_1, \dots, c_r),$$
$$\lambda'_{(2)} = (c_{r+1}, \dots, c_s),$$

$$\mu'_{(1)} = (d_1, \ldots, d_r),$$
$$\mu'_{(2)} = (d_{r+1}, \ldots, d_t).$$

Then

$$[S^{\lambda'} : D^{\mu'}] = [S^{\lambda'_{(1)}} : D^{\mu'_{(1)}}][S^{\lambda'_{(2)}} : D^{\mu'_{(2)}}].$$

Proof. See [5]. □

The method below is derived from Kleshchev's Branching Theorem. (See [18], [10].)

Theorem 4.3.12. *Suppose that μ has exactly m normal r-nodes and let $\overline{\mu}$ be the partition obtained from μ by removing these nodes. Also, let Ω denote the set of partitions of $n - m$ which are obtained from λ by removing m r-nodes. Then Kleschev's Branching Theorem shows that*

$$[S^{\lambda} : D^{\mu}] \leq \sum_{\omega \in \Omega} [S^{\omega} : D^{\overline{\mu}}] \tag{4.3}$$

(Here we interpret the right hand side to be zero when Ω is empty.)

Remark.

1. We may repeat this process. First remove all the m_1 normal r_1-nodes from the abacus display of μ, then remove all the m_2 normal r_2-nodes from the abacus display of $\overline{\mu}$, and so on, until we reach a stage where we can evaluate the decomposition numbers on the right hand side of the inequality.

2. The weight of the partitions involved does not increase when we apply this theorem, i.e. the weight of $\overline{\mu}$ is at most the weight of μ. The reason is as follows. First notice that, in general, if the number of beads on runner $r - 1$ is a and the number of beads on runner r is b, then moving a bead to the

left, from runner r to runner $r-1$, decreases the weight by $a-b+1$ (a negative decrease correspond to an increase). Hence, by induction, moving m beads to the left, from runner r to runner $r-1$, decreases the weight by $m(a-b+m)$. Now suppose that μ has exactly m normal r-nodes. Then the definition of normal nodes implies that $m \geq b-a$; thus, $m(a-b+m) \geq 0$. So, removing the m normal r-nodes does not increase the weight.

Theorem 4.3.13. *Assume that λ is p-regular and we know the decomposition numbers for every S^ν with $\nu \trianglerighteq \lambda$. Then we can express D^λ as a linear combination of the Specht modules S^ν with ν. For all r, the r-restriction of this linear combination of Specht modules is a module for $\mathfrak{S}_{n-1}$.*

Theorem 4.3.14. *[3, 6] Assume that λ and μ are partitions of n with μ being p-regular. Then $[S^\lambda : D^\mu] = [S^{\lambda'} : D^{\mu^*}]$, where λ' is the conjugate of λ and μ^* is the image of μ under the Mullineux map.*

We will give some examples on how to apply some of the methods mentioned above.

Example 4.3.15. *Suppose that we are in the case 4, with $i = p$ and $j = k = p+1$. For simplicity, we take $p = 5$. Thus the partitions belong to block of $k\mathfrak{S}_{27}$ and whose p-core has the following abacus display:*

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & - & - & \bullet \\
- & - & - & - & \bullet
\end{array}$$

The exceptional partitions for this p-core are: $\alpha = \langle p^3 \rangle = \langle 5^3 \rangle$, $\beta = \langle p^2, p-1 \rangle = \langle 5^2, 4 \rangle$, $\gamma = \langle p-1, p \rangle = \langle 4, 5 \rangle$ and $\delta = \langle p-1 \rangle = \langle 4 \rangle$. We will show that part of the decomposition matrix is as follows:

	$\langle p\rangle$	$\langle p^2\rangle$	$\langle p,p-1\rangle$	α	β	γ	δ
$\langle p\rangle$	1						
$\langle p^2\rangle$		1					
$\langle p,p-1\rangle$	1	1	1				
$\alpha=\langle p^3\rangle$				1			
$\beta=\langle p^2,p-1\rangle$		1		1	1		
$\gamma=\langle p-1,p\rangle$		1	1	1	1	1	
$\delta=\langle p-1\rangle$				1	1	1	1

(Note: Blank entries are zero)

Proof. It is easy to check that the set of partitions which index the rows of this matrix are precisely the partitions λ such that λ has the same p-core as δ and $\lambda \trianglerighteq \delta$. Suppose that λ, $\mu \in \{\langle p\rangle, \langle p^2\rangle, \langle p,p-1\rangle\}$. Then $p-1$ applications of Theorem 4.3.6 allow us to equate $[S^\lambda : D^\mu]$ with a decomposition number in case 2 with $i=p$, $j=k=p+1$. Alternatively, we can also apply Schaper's Theorem (4.3.7).

We give some examples on how to calculate $[S^\lambda : D^\mu]$ using Schaper's Theorem. We calculate $[S^\lambda : D^\mu] = [S^{\langle p^2\rangle} : D^{\langle p\rangle}]$.

For $\lambda = \langle p^2\rangle$, we have the abacus display

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & - & - & - \\
- & - & - & - & \bullet \\
- & - & - & - & - \\
- & - & - & - & \bullet
\end{array}$$

We have $\nu_p(i) = 0$ for every i.

For $i = 1$

$$\tau = \begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & - & - & \bullet \\
- & - & - & - & - \\
- & - & - & - & - \\
- & - & - & - & \bullet
\end{array}$$

and

$$\sigma = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ - & - & - & - & \bullet \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & \bullet \end{array}$$

The number of beads crossed by the bead moved to obtain τ from the abacus display of λ is 0.

The number of beads crossed by the bead moved to obtain σ from the abacus display of τ is 0.

The number of beads crossed by the bead moved to obtain τ is of the same parity from that to obtain σ, so we have $(-1)^{\tau\sigma} = -1$. We also have $d_{\sigma\mu} = 1$ because $\sigma = \mu$ thus $[S^\sigma : D^\mu] = 1$. Therefore

$$(-1)^{\tau\mu} d_{\sigma\mu} = -1.$$

For

$$\tau = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ - & - & - & - & - \\ - & - & - & - & \bullet \\ - & - & - & - & \bullet \end{array},$$

there is no $\sigma \in \mathcal{P}$ such that $\sigma > \lambda$.

For $i = 2$, there is no $\lambda \in \mathcal{P}(n - ip)$.

For $i = 3$

$$\tau = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ - & - & - & - & \bullet \\ - & - & - & - & \bullet \\ - & - & - & - & - \end{array},$$

we have

$$\sigma = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ - & - & - & - & \bullet \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & \bullet \end{array}.$$

The number of beads crossed by the bead moved to obtain τ from the abacus display of λ is 1.

The number of beads crossed by the bead moved to obtain σ from the abacus display of τ is 0.

The number of beads crossed by the bead moved to obtain τ is of different parity from that to obtain σ, so we have $(-1)^{\tau\sigma} = 1$. We also have $d_{\sigma\mu} = 1$ because $\sigma = \mu$, thus $[S^{\sigma} : D^{\mu}] = 1$.

Therefore

$$(-1)^{\tau\mu} d_{\sigma\mu} = 1.$$

From Equation 4.1, we have $b_{\lambda\mu} = 0$, so $d_{\lambda\mu} = [S^{\lambda} : D^{\mu}] = [S^{\langle p^2 \rangle} : D^{\langle p \rangle}] = 0$.

With similar arguments, we find that $b_{\lambda\mu} = 1$ for $\lambda = \langle p, p-1 \rangle$ and $\mu \in \{\langle p^2 \rangle, \langle p \rangle\}$. So $d_{\lambda\mu} = 1$.

From Schaper's Theorem, we know that S^{α} is irreducible, so it only has D^{α} as its composition factor.

Using Schaper's Theorem, we also will obtain $[S^{\beta} : D^{\langle p \rangle}] = [S^{\beta} : D^{\langle p, p-1 \rangle}] = 0$ and $[S^{\beta} : D^{\langle p^2 \rangle}] = 1$.

But for $\lambda = \gamma$ and $\mu = \langle p^2 \rangle$, we get $b_{\lambda\mu} = 2$, so either $d_{\lambda\mu} = [S^{\lambda} : D^{\mu}] = 1$ or $d_{\lambda\mu} = [S^{\lambda} : D^{\mu}] = 2$.

We note that our block B forms a $[3:2]$-pair with block $\widetilde{B}$ whose p-core has the following abacus display

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ - & - & - & \bullet & - \\ - & - & - & \bullet & - \end{array}.$$

We now calculate $[S^{\widetilde{\alpha}} : D^{\widetilde{\mu}}]$, where $\widetilde{\alpha} = \langle 5 \rangle_{\widetilde{B}}$ and $\widetilde{\mu} = \langle 4, 4 \rangle_{\widetilde{B}}$. By using Theorem 4.3.7, we have $0 < [S^{\widetilde{\alpha}} : D^{\widetilde{\mu}}] \leq 2$. By repeated use of Theorem 4.3.12, we have the following abacus displays:

$$\breve{\lambda} = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ - & - & - & - & - \\ - & - & - & - & \bullet \end{array} \quad \text{and} \quad \breve{\alpha} = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ - & - & - & - & - \\ - & - & - & - & \bullet \end{array}.$$

Hence, we obtain $[S^{\widetilde{\alpha}} : D^{\widetilde{\mu}}] = 1$. From Theorem 4.3.7, we see that $[S^{\widetilde{\beta}} : D^{\widetilde{\mu}}] = [S^{\widetilde{\gamma}} : D^{\widetilde{\mu}}] = 0$. With the similar argument used for calculating $[S^{\widetilde{\alpha}} : D^{\widetilde{\mu}}]$, we obtain $[S^{\widetilde{\delta}} : D^{\widetilde{\mu}}] = 1$.

Recall equations (A1)-(A4) and (B1)-(B4) in Section 3.4.2. From equation (A1), $[S^{\alpha}\downarrow_{\widetilde{B}} : D^{\widetilde{\lambda}}] = 2$. But $[S^{\alpha}\downarrow_{\widetilde{B}} : D^{\widetilde{\lambda}}] = 2 \times [S^{\alpha} : D^{\lambda}] + [D^{\alpha}\downarrow_{\widetilde{B}} : D^{\widetilde{\lambda}}]$. Hence $[D^{\alpha}\downarrow_{\widetilde{B}} : D^{\widetilde{\lambda}}] = 2$. Now, we consider equation (A3). We have $[S^{\gamma}\downarrow_{\widetilde{B}} : D^{\widetilde{\lambda}}] = 4$. But $[S^{\gamma}\downarrow_{\widetilde{B}} : D^{\widetilde{\lambda}}] = 2 \times [S^{\gamma} : D^{\lambda}] + [D^{\alpha}\downarrow_{\widetilde{B}} : D^{\widetilde{\lambda}}]$. This gives $[S^{\gamma} : D^{\lambda}] = 1$. Using Schaper's Theorem, we have $[S^{\delta} : D^{\langle p \rangle_B}] = [S^{\delta} : D^{\langle p^2 \rangle_B}] = [S^{\delta} : D^{\langle p,p-1 \rangle_B}]$.

From Proposition 3.4.12, we have

$$[S^{\beta} : D^{\alpha}] = [S^{\gamma} : D^{\alpha}] = [S^{\delta} : D^{\alpha}] = 1. \tag{4.4}$$

From Proposition 3.4.13, it is easy to see that:

1. $[S^{\gamma} : D^{\beta}] = 1$.
2. $[S^{\delta} : D^{\beta}] = 1$.
3. $[S^{\delta} : D^{\gamma}] = 1$.

□

4.4 Decomposition Numbers for Blocks of Weight 3

In this section we will find some decomposition numbers for blocks of weight 3 in case 2, 3 and 4.

4.4.1 Some decomposition numbers in case 2.

Assume that the p-core belongs to case 2. So the abacus display has a following property: runners 1 up to $i-1$ each contains 3 beads; runners i up to $j-1$ each contains 4 beads; after this, there are some or no runners with 4 beads. The remaining runners each contain 3 beads. Let runner j be the first runner with 3 beads after runner i.

By Theorem 4.3.6, we can equate the decomposition number $[S^\lambda : D^\mu]$ with a composition number of weight 3 in a smaller block of symmetric group for all partitions λ and μ except when λ is one of the partitions $\alpha^\heartsuit$, $\beta^\heartsuit$, $\gamma^\heartsuit$, $\alpha^\spadesuit$, $\beta^\spadesuit$, $\gamma^\spadesuit$, α^u, β^u, γ^u where

$$\alpha^\heartsuit = \langle i^2 \rangle,\ \beta^\heartsuit = \langle i, i-1 \rangle,\ \gamma^\heartsuit = \langle i-1 \rangle$$

,

$$\alpha^\spadesuit = \langle i^3 \rangle,\ \beta^\spadesuit = \langle (i-1)^2, i \rangle,\ \gamma^\spadesuit = \langle (i-1)^2 \rangle$$

,

$$\alpha^u = \langle i^2, u \rangle, \quad \text{for } 1 \leq u \leq p \text{ and } u \neq i-1, i$$

. (Note: if $u = i-1$, we can use Theorem 4.3.6, and $u = i$ gives $\alpha^\spadesuit$.)

$$\beta^u = \langle i-1, i, u \rangle, \quad \text{for } 1 \leq u \leq p \text{ and } u \neq i-1, i.$$

(Note: if $u = i$, we can use Theorem 4.3.6, and $u = i-1$ gives $\beta^\spadesuit$.)

$$\gamma^u = \langle i-1, u \rangle, \quad \text{for } 1 \leq u \leq p \text{ and } u \neq i-1, i.$$

(Note: if $u = i$, we can use Theorem 4.3.6, and $u = i-1$ gives $\gamma^\spadesuit$.)
The partitions $\alpha^\heartsuit$, $\beta^\heartsuit$, $\gamma^\heartsuit$, $\alpha^\spadesuit$, $\beta^\spadesuit$, $\gamma^\spadesuit$, α^u,β^u and γ^u are called the exceptional partition for case 2. They have the abacus displays as follows

$$\alpha^\heartsuit = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & - & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & \bullet & - & - & - & - & - & - & - & - & -
\end{array}$$

so, $\alpha^\heartsuit = (2p - j + 2i, i^{j-i}, 1^{p-i})$;

$$\beta^\heartsuit = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & \bullet & - & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & \bullet & - & - & - & - & - & - & - & - & -
\end{array}$$

so $\beta^\heartsuit = (2p - j + 2i, i^{j-i-1}, i-1, 1^{p-i+1})$;

$$\gamma^\heartsuit = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & - & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & \bullet & - & - & - & - & - & - & - & - & - & -
\end{array}$$

so $\gamma^{\heartsuit} = (2p - j + 2i - 1, i^{j-i}, 1^{p-i+1})$.

$$\alpha^{\spadesuit} = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & - & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - \\
- & - & - & \bullet & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & -
\end{array}$$

so $\alpha^{\spadesuit} = (p - j + 2i, i^{j-i}, 1^{2p-i})$;

$$\beta^{\spadesuit} = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & \bullet & - & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - \\
- & - & - & \bullet & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & -
\end{array}$$

so $\beta^{\spadesuit} = (p - j + 2i, i^{j-i-1}, i - 1, 1^{2p-i+1})$;

$$\gamma^{\spadesuit} = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & - & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - \\
- & - & \bullet & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & -
\end{array}$$

so $\gamma^{\spadesuit} = (p - j + 2i - 1, i^{j-i}, 1^{2p-i+1})$.

$$\alpha^{u} = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & - & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - \\
- & - & - & \bullet & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & -
\end{array} \ + \text{one more `gap' on runner } u\,,$$

so

$$\alpha^{u} = \begin{cases} (p - j + 2i, u - j + i + 1, (i + 1)^{j-i}, 2^{p-u}, 1^{u-i-1}), & \text{if } j \leq u \leq p, \\ ((p - j + i + u, p - j + 2i + 1, (i + 1)^{j-u-1}, i^{u-i}, 1^{p-1}), & \text{if } i < u < j, \\ (p - j + 2i, i^{j-i}, u + 1, 2^{p-i}, 1^{i-u-1}), & \text{if } 1 \leq u < i - 1; \end{cases}$$

$$\beta^{u} = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & \bullet & - & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - \\
- & - & - & \bullet & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & -
\end{array} \ + \text{one more `gap' on runner } u,$$

so

$$\beta^{u} = \begin{cases} (p - j + 2i, u - j + i + 1, (i + 1)^{j-i-1}, i, 2^{p-u}, 1^{u-i}), & \text{if } j \leq u \leq p, \\ ((p - j + i + u, p - j + 2i + 1, (i + 1)^{j-u-1}, i^{u-i-1}, i - 1, 1^{p-i+1}), & \text{if } i < u < j, \\ (p - j + 2i, i^{j-i-1}, i - 1, u + 1, 2^{p-i+1}, 1^{i-u-2}), & \text{if } 1 \leq u < i - 1; \end{cases}$$

$$\gamma^u = \begin{array}{ccccccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ - & - & - & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - \\ - & - & \bullet & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - & - \end{array} \quad + \text{one more `gap' on runner } u,$$

so

$$\gamma^u = \begin{cases} (p-j+2i-1, u-j+i+1, (i+1)^{j-i}, 2^{p-u}, 1^{u-i}), & \text{if } j \leq u \leq p, \\ ((p-j+i+u, p-j+2i, (i+1)^{j-u-1}, i^{u-i}, 1^{p-i+1}), & \text{if } i < u < j, \\ (p-j+2i-1, i^{j-i}, u+1, 2^{p-i+1}, 1^{i-u-2}), & \text{if } 1 \leq u < i-1; \end{cases}$$

Remark. The three partitions $\alpha^{\spadesuit}$, $\beta^{\spadesuit}$ and $\gamma^{\spadesuit}$ are p-singular and $\alpha^{\spadesuit} \rhd \beta^{\spadesuit} \rhd \gamma^{\spadesuit}$ and $\alpha^{\heartsuit} \rhd \beta^{\heartsuit} \rhd \gamma^{\heartsuit}$ and $\alpha^u \rhd \beta^u \rhd \gamma^u$, for all u with $1 \leq u \leq p$ and $u \neq i-1, i$. If $\lambda \in \{\alpha, \beta, \gamma\}$ then

$$\lambda^{\heartsuit} \rhd \lambda^{j-1} \rhd \lambda^{j-2} \rhd \cdots \rhd \lambda^{i+1} \rhd \lambda^{p} \rhd \lambda^{p-1} \rhd \cdots \rhd \lambda^{j} \rhd \lambda^{i-2} \rhd \lambda^{i-3} \rhd \cdots \rhd \lambda^{1} \rhd \lambda^{\spadesuit}. \quad (4.5)$$

Proposition 4.4.1. *For $1 \leq u \leq p$ and $u \neq i-1, i$ and for:*

1. *For arbitrary λ and $\mu \in \{\alpha^{\heartsuit}, \alpha^u\}$, we can determine $[S^\lambda : D^\mu]$.*

2. *For λ an exceptional partition and $\mu \in \{\beta^{\heartsuit}, \beta^u\}$, we can determine $[S^\lambda : D^\mu]$.*

3. *For λ an exceptional partition and $\mu \in \{\gamma^{\heartsuit}, \gamma^u\}$, we can determine $[S^\lambda : D^\mu]$.*

Proof. We will show that we can reduce the calculation of the decomposition numbers in the proposition to the cases of calculating the decomposition numbers for blocks of weight $0, 1,$ and 2. We recall that all the decomposition numbers for any block of weight 0, 1 and 2 are already fully determined. There are some cases to be considered. For easy visualization of the methods being used, we give an example, and the reader can easily make a generalization. Take $i = 4$, $j = 9$, $p = 13, u = 13$.

1. Suppose that $\mu \in \{\alpha^{\heartsuit}, \alpha^u\}$. Then μ has 2 normal 4-nodes and every λ in the same block as μ has at most 2 removable 4-nodes. Hence, we can apply

Proposition 4.3.1 to compute $[S^\lambda : D^\mu]$.

$$\alpha^\heartsuit = \begin{array}{ccccccccccccc} \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&-&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ -&-&-&\bullet&\bullet&\bullet&\bullet&\bullet&-&-&-&-&- \\ -&-&-&-&-&-&-&-&-&-&-&-&- \\ -&-&-&\bullet&-&-&-&-&-&-&-&-&- \end{array} \qquad \alpha^u = \begin{array}{ccccccccccccc} \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&-&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&- \\ -&-&-&\bullet&\bullet&\bullet&\bullet&\bullet&-&-&-&-&\bullet \\ -&-&-&\bullet&-&-&-&-&-&-&-&-&- \\ -&-&-&-&-&-&-&-&-&-&-&-&- \end{array}$$

.

2. Suppose that $\mu \in \{\beta^\heartsuit, \beta^u\}$. If $i \neq 2$ and $\mu \neq \beta^{i-2}$ then μ has exactly one normal $(i-1)$-node, and λ has at most 1 removable $(i-1)$-node, so again, we can apply Proposition 4.3.1.

   Now, assume that $u = \beta^{i-2}$ and $i \neq 2, 3$. then μ has exactly one normal $(i-2)$-node, and λ has at most 1 removable $(i-2)$-node, so we can apply Proposition 4.3.1.

$$\mu = \beta^2 = \begin{array}{ccccccccccccc} \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&-&-&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ -&\bullet&\bullet&-&\bullet&\bullet&\bullet&\bullet&-&-&-&-&- \\ -&-&-&\bullet&-&-&-&-&-&-&-&-&- \\ -&-&-&-&-&-&-&-&-&-&-&-&- \end{array}$$

   Note that if $\mu = \beta^{i-2}$ and $i = 3$, then μ is p-singular.

   Assume that $i = 2$ and μ is p-regular. Then $\mu = \beta^u$ for some u with $j \leq u \leq p$. We need only consider those partitions λ for which the first part of μ is larger that the first part of λ, since otherwise, either $\mu \not\trianglerighteq \lambda$ or we can apply Theorem 4.3.10. Therefore, $\lambda \in \{\gamma^\spadesuit, \gamma^j, \ldots, \gamma^p\}$. But notice that β^u and γ^u have the same first column length and thus we can apply Theorem 4.3.8. We see that β^u has a normal u-node while $\gamma^\spadesuit$ has no removable u-node; unless $u = j$. So, $[S^\lambda : D^\mu] = 0$, by Proposition 4.3.1.

   There is one last case to consider, namely $i = 2$, $\mu = \beta^j$ (with $j \leq p$, since otherwise μ is singular) and $\lambda = \gamma^\spadesuit$. Suppose $u = 9$, $j = 9$, $p = 13$. We have the following abacus displays

$$\mu = \beta^j = \beta^9 = \langle 9, 2, 1 \rangle = \begin{array}{ccccccccccccc} \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ -&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&-&\bullet&\bullet&\bullet&\bullet \\ \bullet&-&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&-&-&-&- \\ -&\bullet&-&-&-&-&-&-&-&-&-&-&- \\ -&-&-&-&-&-&-&-&-&-&-&-&- \end{array}$$

$$\lambda = \gamma^{\spadesuit} = \langle 1, 1\rangle = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - \\
\bullet & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & -
\end{array}$$

In order to handle this case, we need the following relationship, drawn from the remark in ([7], page 84) and also in [13].

Let μ be a partition of $n - p$. Then

$$\sum_{\nu} \pm S^{\nu} = 0, \tag{4.6}$$

where the sum is over all partitions ν where $[\nu]$ is obtained by wrapping a rim p-hook onto $[\mu]$, and the sign is positive if and only if the leg length of the wrapped-on skew hook is even.

Let μ be the partition of 33 and has the following abacus display

$$\mu = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - \\
\bullet & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & -
\end{array}$$

We now add a rim p-hook to μ in all possible ways, using equation 4.6, to obtain the following relationship:

$$S^{\langle 1^2\rangle} - S^{\langle 1,9\rangle} + S^{\langle 1,10\rangle} - S^{\langle 1,11\rangle} + S^{\langle 1,12\rangle} - S^{\langle 1,13\rangle} - S^{\langle 1,2\rangle} +$$
$$S^{\langle 1,3\rangle} - S^{\langle 1,4\rangle} + S^{\langle 1,5\rangle} - S^{\langle 1,6\rangle} + S^{\langle 1,7\rangle} - S^{\langle 1,8\rangle} + S^{\langle 1\rangle} = 0,$$

where $\gamma^{\spadesuit} = \langle 1^2\rangle$. So, $[S^{\gamma^{\spadesuit}} : D^{\beta^j}]$ is equal to the multiplicity of D^{β^j} in

$$S^{\langle 1,9\rangle} - S^{\langle 1,10\rangle} + S^{\langle 1,11\rangle} - S^{\langle 1,12\rangle} + S^{\langle 1,13\rangle} + S^{\langle 1,2\rangle} -$$
$$S^{\langle 1,3\rangle} + S^{\langle 1,4\rangle} - S^{\langle 1,5\rangle} + S^{\langle 1,6\rangle} - S^{\langle 1,7\rangle} + S^{\langle 1,8\rangle} - S^{\langle 1\rangle}$$

However, since β^j does not dominate all partitions involved except $\langle 1, 9\rangle$, the multiplicity of D^{β^j} in $S^{\gamma^{\spadesuit}}$ is equal to $[S^{\langle 1,9\rangle} : D^{\beta^j}]$. Using Theorem 4.3.8 twice and the defect 1 result, we obtain $[S^{\langle 1,9\rangle} : D^{\beta^j}] = 1$. Hence, $[S^{\gamma^{\spadesuit}} : D^{\beta^j}] = 1$. Alternatively, we can use Theorem 4.3.7 to derive the same result.

3. Suppose that $\mu \in \{\gamma^{\heartsuit}, \gamma^{u}\}$. Note that μ has exactly one normal $(i-1)$ node (except if $i = 2$, $j = p+1$ and $\mu = \gamma^u$), but every exceptional partition λ has at most 1 removable $(i-1)$-node, so we can apply Proposition 4.3.1 again. Suppose that $i = 2$, $j = p+1$ and $\mu = \gamma^u$ (where $i < u < p$). We need only consider those λ where the first part of μ exceeds the first part of λ, and λ has a removable u-node. It is easy to check that indeed there is no such partition, so we are done.

$\square$

4.4.2 Some decomposition numbers in case 3.

As usual, we will take a specific example for simplicity. Let $i = 3$, $j = 8$, $l = 11$ and $p = 13$. Using Theorem 4.3.6, we can equate the decomposition number $[S^\lambda : D^\mu]$ with a decomposition number of weight 3 in a smaller symmetric group for all partitions λ and μ in the block, except when $\lambda \in \{\alpha, \beta, \gamma, \delta\}$, where

$$\alpha = \langle i^2, i+1\rangle = \langle 3,3,4\rangle,\ \beta = \langle i-1, i, i+1\rangle = \langle 2,3,4\rangle,\ \gamma = \langle i^2\rangle = \langle 3^2\rangle, \text{ and}$$

$$\delta = \langle i, i-1\rangle = \langle 3,2\rangle$$

We will call α, β, γ and δ as the exceptional partitions for case 3. The abacus displays for the exceptional partitions in case 3 are as follows:

$$\alpha = \langle 3,3,4\rangle = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - \\
- & - & \bullet & - & \bullet & \bullet & \bullet & - & - & - & - & - & - \\
- & - & - & \bullet & - & - & - & - & - & - & - & - & -
\end{array}$$

So $\alpha = (2p-l-j+3i+2, (p-l+2i+1)^{j-i-2}, p-l+2i, i^{l-i}, 1^{p-i}) = (18, 9^3, 8, 3^8, 1^{10})$;

$$\beta = \langle 2,3,4\rangle = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - \\
- & - & \bullet & - & \bullet & \bullet & \bullet & - & - & - & - & - & - \\
- & - & - & \bullet & - & - & - & - & - & - & - & - & -
\end{array}$$

So $\beta = (2p-l-j+3i+2, (p-l+2i+1)^{j-i-2}, p-l+2i, i^{l-i-1}, i-1, 1^{p-i+1}) = (18, 9^3, 8, 3^7, 2, 1^{11})$;

$$\gamma = \langle 3,3\rangle = \begin{matrix} \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&-&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ -&-&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&-&-&- \\ -&-&-&\bullet&\bullet&\bullet&\bullet&-&-&-&-&-&- \\ -&-&\bullet&-&-&-&-&-&-&-&-&-&- \end{matrix}$$

So $\gamma = (2p-l-j+3i+1, (p-l+2i+1)^{j-i-1}, i^{l-i}, 1^{p-i} = (17, 9^4, 3^8, 1^{10})$;

$$\delta = \langle 3,2\rangle = \begin{matrix} \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&-&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ -&\bullet&-&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&-&-&- \\ -&-&-&\bullet&\bullet&\bullet&\bullet&-&-&-&-&-&- \\ -&-&\bullet&-&-&-&-&-&-&-&-&-&- \end{matrix}$$

So $\delta = (2p-l-j+3i+1, (p-l+2i+1)^{j-i-1}, i^{l-i-1}, i-1, 1^{p-i+1}) = (17, 9^4, 3^7, 2, 1^{11})$.

Proposition 4.4.2. *The part of the decomposition matrix whose rows and columns are labeled by* α, β, γ *and* δ *is*

	α	β	γ	δ
α	1			
β	1	1		
γ	1		1	
δ	1	1	1	1

.

(Note: Blank entries are zero)

Proof. Note that $\alpha \rhd \beta \rhd \delta$ and $\alpha \rhd \gamma \rhd \delta$ but $\beta \not\rhd \gamma$ and $\gamma \not\rhd \beta$. First, we calculate $[S^\beta : D^\alpha]$. By repeated use of Theorem 4.3.10, i.e. removing the last 5 beads of the abacus displays of the partitions, we get the partitions of block of defect 1. Recalling the explanation in Section 3.2, we have $[S^\beta : D^\alpha] = 1$. Alternatively, we can use Theorem 4.3.9, with $\alpha_{(1)} = (18, 9^3, 8)$ and $\beta_{(1)} = (18, 9^3, 8)$ and $\alpha_{(2)} = (3^8, 1^10)$ and $\beta_{(2)} = (3^7, 2, 1^11)$. It is to see that $[S^{\beta_{(1)}} : D^{\alpha_{(1)}}] = [S^{\beta_{(2)}} : D^{\alpha_{(2)}}] = 1$, and again we may conclude that $[S^\beta : D^\alpha] = 1$.

To calculate $[S^\gamma : D^\alpha]$, we use Theorem 4.3.8. We put beads in the first 8 gaps of the abacus displays of γ and α, and again we have the partitions of block of defect 1. Following the same argument as above, we may conclude that $[S^\gamma : D^\alpha] = 1$.

Since $\beta \not\trianglerighteq \gamma$, we have $[S^\gamma : D^\beta] = 0$.

To show that $[S^\delta : D^\gamma] = 1$, we use Theorem 4.3.10. We remove the last 12 beads in the abacus displays of δ and γ. With similar argument as above, we get $[S^\delta : D^\gamma] = 1$. We also can use Theorem 4.3.9 to obtain the same result.

To calculate $[S^\delta : D^\beta]$, we use Theorem 4.3.8. We put beads in the first 8 gaps of the abacus displays of δ and β. Following the same argument as above, we may conclude that $[S^\delta : D^\beta] = 1$.

We show that $[S^\delta : D^\alpha] = 1$. We will use Theorem 4.3.9. First observe that for $r = 5$, we have $\delta_{(1)} = (17, 9^4)$ and $\alpha_{(1)} = (18, 9^3, 8)$. We also have $\delta_{(2)} = (3^7, 2, 1^1 1)$ and $\alpha_{(2)} = (3^8, 1^1 0)$. By explanation in Section 3.2, we see that $[S^{\delta_{(1)}} : D^{\alpha_{(1)}}] = [S^{\delta_{(2)}} : D^{\alpha_{(2)}}] = 1$, hence we obtain $[S^\delta : D^\alpha] = 1$. $\square$

4.4.3 Some decomposition numbers in Case 4.

Now, we let $i = 3$, $j = 8$ and $l = 11$. Using Theorem 4.3.6, we can equate the decomposition number $[S^\lambda : D^\mu]$ with a decomposition number of weight 3 in a smaller symmetric group for all partitions λ and μ in the block, except when $\lambda \in \{\alpha, \beta, \gamma, \delta\}$, where

$$\alpha = \langle i^3 \rangle,\ \beta = \langle i^2, i-1 \rangle,\ \gamma = \langle i-1, i \rangle,\ \text{and}\ \delta = \langle i-1 \rangle,$$

We will call α, β, γ and δ as the exceptional partitions for case 4. The abacus displays for the exceptional partitions in case 4 are as follows:

$$\alpha = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - \\
- & - & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - & - \\
- & - & \bullet & - & - & - & - & - & - & - & - & - & -
\end{array}$$

So $\alpha = (2p-l-j+3i, (p-l+2i)^{j-i}, i^{l-i}, 1^{p-i}) = (16, 8^5, 3^8, 1^{10})$;

$$\beta = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - \\
- & - & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - & - \\
- & - & \bullet & - & - & - & - & - & - & - & - & - & -
\end{array}$$

So $\beta = (2p-l-j+3i, (p-l+2i)^{j-i}, i^{l-i-1}, i-1, 1^{p-i+1}) = (16, 8^5, 3^7, 2, 1^{11})$;

$$\gamma = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - \\
- & \bullet & - & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - & - \\
- & - & \bullet & - & - & - & - & - & - & - & - & - & -
\end{array}$$

So $\gamma = (2p-l-j+3i, (p-l+2i)^{j-i-1}, p-l+2i-1, i^{l-i}, 1^{p-i+1}) = (16, 8^4, 7, 3^8, 1^{11})$;

$$\delta = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - \\
- & - & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - & - \\
- & \bullet & - & - & - & - & - & - & - & - & - & - & -
\end{array}$$

So $\delta = (2p-l-j+3i-1, (p-l+2i)^{j-i}, i^{l-i}, 1^{p-i+1}) = (15, 8^5, 3^8, 1^{11})$.

Proposition 4.4.3. *The part of the decomposition matrix whose rows and columns are labeled by α, β, γ and δ is*

	α	β	γ	δ
α	1			
β	1	1		
γ	1	1	1	
δ	1	1	1	1

.

(Note: Blank entries are zero).

Proof. This follows directly from Proposition 3.4.12 and Proposition 3.4.13 as this block forms a [3 : 2]-pair with block $\widetilde{B}$ where the abacus display of its p-core is

$$\begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
- & \bullet & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & - & - & - \\
- & \bullet & - & \bullet & \bullet & \bullet & \bullet & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & -
\end{array}$$

$\square$

Chapter 5

Decomposition Numbers of a Rouquier Block and its [3 : 2]-pair

5.1 The Rouquier Core and Rouquier Block

Given a p-core κ and a nonnegative integer ω, consider partitions with p-core κ and p-weight ω. Choosing m so that any such partition has less than or equal to m nonzero parts, and representing these partitions on an abacus with m beads, we have a p-quotient for each partition. This gives a bijection between this set of partitions and the set of p-tuples $(\delta^{(1)}, \ldots, \delta^{(p)})$ of partitions satisfying $|\delta^{(1)}| + \ldots + |\delta^{(p)}| = \omega$, where $|\delta^{(i)}|$ denote the weight of each runner of the abacus display of the associated partition.

Let ρ be a p-core which satisfies the following property: ρ has an abacus display in which each runner other than the first (leftmost) runner has at least $\omega - 1$ more beads than the runner to its immediate left. After choosing such a ρ, we may assume that there are at least ω beads on each runner. Let m be the number of beads in such an abacus display of ρ. One possible choice for ρ is the p-core which has an abacus display with $\omega + (i-1)(\omega - 1)$ beads on the i-th runner, for

$i = 1, \ldots, p$. We will choose this p-core. This p-core and its corresponding blocks are called the *Rouquier core* and *Rouquier blocks* respectively.

We note that the partition λ belonging to this block, whose p-quotient is $(\lambda^{(1)}, \ldots, \lambda^{(p)})$, is p-regular if, and only if, $\lambda^{(1)}1 = \emptyset$.

We take note that the Rouquier block B only forms $[3:2]$-pairs with some blocks $\widetilde{B_i}$ of $k\mathfrak{S}_{n-2}$ and no $[3:1]$-pairs with any block of $k\mathfrak{S}_{n-1}$. The subscript i refers to runner i which has 2 more beads than runner $i-1$, i.e. the abacus display of block $\widetilde{B_i}$ is obtained by interchanging runners i and $i-1$ of the abacus display of block B.

From now on, we will focus on a Rouquier block with $p = 5$ and $\omega = 3$ in $k\mathfrak{S}_{135}$. The abacus display of its p-core is as follow:

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
- & \bullet & \bullet & \bullet & \bullet \\
- & \bullet & \bullet & \bullet & \bullet \\
- & - & \bullet & \bullet & \bullet \\
- & - & \bullet & \bullet & \bullet \\
- & - & - & \bullet & \bullet \\
- & - & - & \bullet & \bullet \\
- & - & - & - & \bullet \\
- & - & - & - & \bullet
\end{array}$$

This block makes a $[3:2]$-pair with a block $\widetilde{B}$ of $k\mathfrak{S}_{133}$, which has the following p-core:

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
- & \bullet & \bullet & \bullet & \bullet \\
- & \bullet & \bullet & \bullet & \bullet \\
- & - & \bullet & \bullet & \bullet \\
- & - & \bullet & \bullet & \bullet \\
- & - & - & \bullet & \bullet \\
- & - & - & \bullet & \bullet \\
- & - & - & \bullet & - \\
- & - & - & \bullet & -
\end{array}$$

With respect to $[3:2]$-pair above, B has 61 non-exceptional partitions, which we will order using lexicographical order, and denoted as λ_i, where $i = \{1, \ldots, 61\}$.

These partitions only have two beads that can be moved from runner 5 to runner 4 in their abacus display.

There are four exceptional partitions, namely $\alpha = \langle p^3 \rangle_B = \langle 5^3 \rangle_B$, $\beta = \langle p^2, p - 1 \rangle_B = \langle 5^2, 4 \rangle_B$, $\gamma = \langle p-1, p \rangle_B = \langle 4, 5 \rangle_B$ and $\delta = \langle p - 1 \rangle_B = \langle 4 \rangle_B$.

Among the non-exceptional partitions, there are forty 5-regular partitions, and the rest are 5-singular partitions. So, its decomposition matrix has sixty-five rows and forty columns.

Again, with respect to [3 : 2]-pair above, block $\widetilde{B}$ has sixty-one non-exceptional partitions, which we will order using lexicographical order, and denoted as $\widetilde{\lambda}_i$, where $i = \{1, \ldots, 61\}$.

Their abacus displays is obtained from the abacus displays of λ by moving their two normal beads from runner 5 to runner 4.

There are four exceptional partitions, namely $\widetilde{\alpha} = \langle p\rangle_{\widetilde{B}} = \langle 5\rangle_{\widetilde{B}}$, $\widetilde{\beta} = \langle p, p-1\rangle_{\widetilde{B}} = \langle 5, 4\rangle_{\widetilde{B}}$, $\widetilde{\gamma} = \langle p, (p-1)^2\rangle_{\widetilde{B}} = \langle 5, 4^2\rangle_{\widetilde{B}}$ and $\widetilde{\delta} = \langle (p-1)^3\rangle_{\widetilde{B}} = \langle 4^3\rangle_{\widetilde{B}}$.

5.2 Chuang and Tan's Method

Our last method for computing the decomposition number only works for Rouquier blocks. This method is due to [4].

Suppose λ and μ both belong to a Rouquier block and have p-quotients ($\lambda^{(0)}$, $\lambda^{(1)}$, $\ldots, \lambda^{(p-1)}$) and ($\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(p-1)}$) respectively. For each $\lambda^{(i)}$ we split its diagram into 2 partitions $\alpha^{(i)}$ and $\beta^{(i)}$. We may have more than one possibility of ($\alpha^{(i)}, \beta^{(i)}$-pair. Then we 'add up' the diagram of the conjugate of $\beta^{(i)}$ with the diagram of $\alpha^{(i+1)}$ to obtain a partition $\mu^{(i+1)}$. From [4], we have the following:

$$[S^\lambda : D^\mu] = \sum \left(c^{\lambda^{(0)}}_{(\emptyset)(\beta^0)} c^{\lambda^{(1)}}_{(\alpha^1)(\beta^1)} \cdots c^{\lambda^{(p-2)}}_{(\alpha^{p-1})(\beta^{p-2})} c^{\lambda^{(p-1)}}_{(\alpha^{p-1})(\emptyset)} \right) \times \left(c^{\mu^{(1)}}_{((\beta^0)')(\alpha^1)} c^{\mu^{(2)}}_{((\beta^1)')(\alpha^2)} \cdots c^{\mu^{(p-1)}}_{((\beta^{p-2})')(\alpha^{p-1})} \right) \tag{5.1}$$

where

$$c^\lambda_{\mu\nu} = [(S^\mu \otimes S^\nu)\uparrow^{\mathfrak{S}_n} : S^\lambda]_{\mathbb{Q}}$$

is a Littlewood-Richardson coefficient (see Theorem 2.3.2) and the sum is over all

μ	ν	λ	$c^{\lambda}_{\mu\nu}$
$\emptyset$	$\yng(3)$	$\yng(3)$	1
$\emptyset$	$\yng(1,1,1)$	$\yng(1,1,1)$	1
$\emptyset$	$\yng(2,1)$	$\yng(2,1)$	1
$\yng(1)$	$\yng(2)$	$\yng(2,1)$ and $\yng(3)$	1
$\yng(1)$	$\yng(1,1)$	$\yng(2,1)$ and $\yng(1,1,1)$	1
$\yng(1,1)$	$\yng(1)$	$\yng(2,1)$ and $\yng(1,1,1)$	1
$\yng(2)$	$\yng(1)$	$\yng(2,1)$ and $\yng(3)$	1
$\yng(3)$	$\emptyset$	$\yng(3)$	1
$\yng(1,1,1)$	$\emptyset$	$\yng(1,1,1)$	1
$\yng(2,1)$	$\emptyset$	$\yng(2,1)$	1

Table 5.1:

the possibilities of splitting up and adding up the diagrams of $\lambda^{(i)}$ to obtain the diagram $\mu^{(j)}$. If the weight of the block $\omega \leq 3$, then

1. $c^{\lambda}_{\mu\nu}$ is equal to 0 or 1.

2. If $\omega = 3$, we list all the possibilities of μ, ν and λ for $c^{\lambda}_{\mu\nu}$ equal to 1 in the Table 5.1.

3. There is one and only one way of splitting up the diagram $[\lambda]$ and adding up to obtain $[\mu]$. So the summation in Equation (5.1) has only one factor.

Hence, we may conclude that $[S^{\lambda} : D^{\mu}]$ is either equal to 1 or equal to 0.

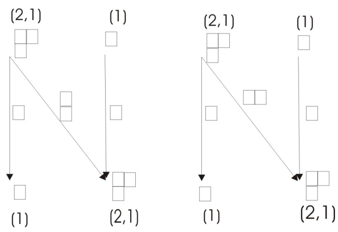


Figure 5.1:

Remark.

1. $\mu^{(0)} = \emptyset$ because μ is p-regular.

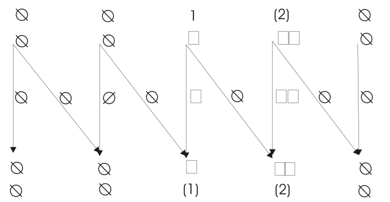
2. If, for example, $\omega = 4$, then there is a possibility that $[S^\lambda : D^\mu] \geq 2$. Consider the situation in figure 5.1. We see that there are 2 possible ways to add up the diagrams to obtain the diagram $[\mu]$. So in this case $[S^\lambda : D^\mu] \geq 2$.

We will not go through the details of this method. Instead we will give an example to show how it works in Section 5.3 below.

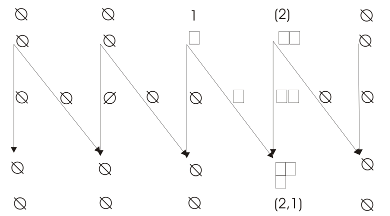
5.3 Decomposition Number of Rouquier Block of $k\mathfrak{S}_{135}$

We are able to calculate all the decomposition numbers $[S^\sigma : D^\lambda]$ of a Rouquier block of $k\mathfrak{S}_{135}$, a block with $p = 5$ and p-weight equal to 3, using the method in Section 5.2.

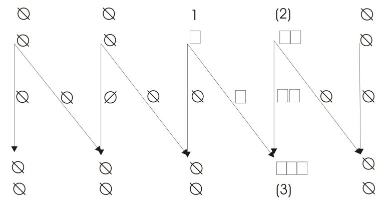
We give an example on how to use this method as shown in the figure below.



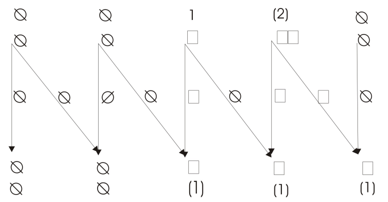
$\mu = <4,3>$



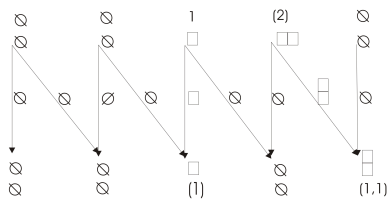
$\mu = <4,4>$



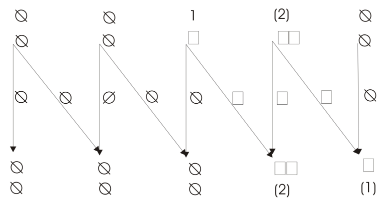
$\mu = <4>$



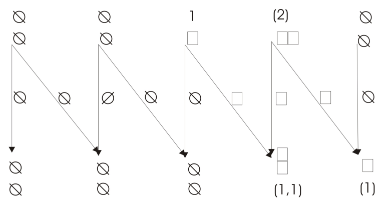
$\mu = <5,4,3>$



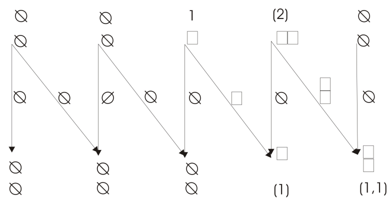
$\mu = <5,5,3>$



μ = <4,5>



μ = <5,4,4>



μ = <5,5,4>

We will find all the decomposition factors of $S^{\langle 4,3\rangle_B}$, all of which have multiplicity 1.

The 5-quotient of $\lambda = \langle 4,3\rangle_B$ is $\{\emptyset, \emptyset, (1), (2), \emptyset\}$. For each $\lambda^{(i)}$, we split it into two possible partitions, $\alpha^{(i)}$ and $\beta^{(i)}$. We then add up the conjugate partition of $\beta^{(i)}$ with $\alpha^{(i+1)}$ to form partition $\mu^{(i+1)}$. There are 8 possible partitions μ, such that D^{μ} is a composition factor of $S^{\langle 4,3\rangle}{}_B$. We list all the possible partitions μ with its 5-quotient as follows:

1. $\mu = \{\emptyset, \emptyset, (1), (2), \emptyset\} = \langle 4,3\rangle_B$
2. $\mu = \{\emptyset, \emptyset, \emptyset, (2,1), \emptyset\} = \langle 4,4\rangle_B$
3. $\mu = \{\emptyset, \emptyset, \emptyset, (3), \emptyset\} = \langle 4\rangle_B$
4. $\mu = \{\emptyset, \emptyset, (1), (1), (1)\} = \langle 5,4,3\rangle_B$
5. $\mu = \{\emptyset, \emptyset, (1), \emptyset, (1,1)\} = \langle 5,5,3\rangle_B$
6. $\mu = \{\emptyset, \emptyset, \emptyset, (2), (1)\} = \langle 4,5\rangle_B$
7. $\mu = \{\emptyset, \emptyset, \emptyset, (1,1), (1)\} = \langle 5,4,4\rangle_B$
8. $\mu = \{\emptyset, \emptyset, \emptyset, (1), (1,1)\} = \langle 5,5,4\rangle_B$

Thus, we have the following result.

$$[S^{\langle 4,3\rangle_B} : D^{\langle 4,3\rangle_B}] = [S^{\langle 4,3\rangle_B} : D^{\langle 4,4\rangle_B}] = [S^{\langle 4,3\rangle_B} : D^{\langle 4\rangle_B}] = [S^{\langle 5,4,3\rangle_B} : D^{\langle 5,5,3\rangle_B}] =$$
$$[S^{\langle 4,3\rangle_B} : D^{\langle 4,5\rangle_B}] = [S^{\langle 4,3\rangle_B} : D^{\langle 5,4,4\rangle_B}] = [S^{\langle 4,3\rangle_B} : D^{\langle 5,5,4\rangle_B}] = 1$$

We calculate all other decomposition numbers in the similar way.

Part of the decomposition matrix is

	$\langle 5\rangle_B$	$\langle 5,5\rangle_B$	$\langle 5,4\rangle_B$	α	β	γ	δ
$\langle 5\rangle_B$	1						
$\langle 5,5\rangle$		1					
$\langle 5,4\rangle$	1	1	1				
$\alpha = \langle 5,5,5\rangle_B$				1			
$\beta = \langle 5,5,4\rangle_B$		1		1	1		
$\gamma = \langle 4,5\rangle_B$		1	1	1	1	1	
$\delta = \langle 4\rangle_B$				1	1	1	1

(Note: Blank entries are zero.)

Notice that we have a similar table as that in example 4.3.15. This is not a mere coincidence. The reason is that we can get the abacus display of the partitions in example 4.3.15 by adding beads into the first 12 gaps of abacus display of these partitions in $k\mathfrak{S}_{135}$. So, for example, $[S^{\langle 5,4\rangle} : D^{\langle 5\rangle}] = 1$ in both block of $k\mathfrak{S}_{135}$ and block of $k\mathfrak{S}_{27}$, by application of Theorem 4.3.8.

Note, however, that we cannot equate each and every decomposition number in both blocks. There are some partitions in block of $k\mathfrak{S}_{135}$ such that when we add beads to its first 12 gaps, its weight is reduced. In such partitions, the decomposition numbers of the entries at the same position of the decomposition matrix can be different. For example, when $\lambda = \langle 5,3\rangle$ and $\mu = \langle 5,4\rangle$, the decomposition number $[S^\lambda : D^\mu] = 1$ in $k\mathfrak{S}_{135}$ but $[S^\lambda : D^\mu] = 0$ in $k\mathfrak{S}_{27}$.

We give the complete decomposition matrix of the block of $k\mathfrak{S}_{135}$ in Appendix A.

5.4 Decomposition Number of block of $k\mathfrak{S}_{133}$

We will show that part of the decomposition matrix is

	$\langle 4\rangle_{\widetilde{B}}$	$\langle 4^2\rangle_{\widetilde{B}}$	$\langle 4,5\rangle_{\widetilde{B}}$	$\widetilde{\alpha}$	$\widetilde{\beta}$	$\widetilde{\gamma}$	$\widetilde{\delta}$
$\langle 4\rangle_{\widetilde{B}}$	1						
$\langle 4^2\rangle_{\widetilde{B}}$		1					
$\langle 4,5\rangle_{\widetilde{B}}$	1	1	1				
$\widetilde{\alpha} = \langle 5\rangle_{\widetilde{B}}$		1	1	1			
$\widetilde{\beta} = \langle 5,4\rangle_{\widetilde{B}}$				1	1		
$\widetilde{\gamma} = \langle 5,4^2\rangle_{\widetilde{B}}$			1	1	1	1	
$\widetilde{\delta} = \langle 4^3\rangle_{\widetilde{B}}$		1	1	1	1	1	1

(Note: Blank entries are zero.)

This $\widetilde{B}$ block of $k\mathfrak{S}_{133}$ forms a $[3:2]$-pair with the Rouquier block of $k\mathfrak{S}_{135}$. Using Lemma 3.4.10 and its remark, we are able to determine all the entries of the decomposition matrix of $\widetilde{B}$ except for the four rows that contain $S^{\widetilde{\alpha}}$, $S^{\widetilde{\beta}}$, $S^{\widetilde{\gamma}}$ and $S^{\widetilde{\delta}}$ respectively.

From Proposition 3.4.12, we have

$$[S^{\widetilde{\alpha}} : D^{\widetilde{\alpha}}] = [S^{\widetilde{\alpha}} : D^{\widetilde{\beta}}] = [S^{\widetilde{\gamma}} : D^{\widetilde{\alpha}}] = [S^{\widetilde{\delta}} : D^{\widetilde{\alpha}}] = 1. \tag{5.2}$$

From Proposition 3.4.13, it is easy to see that:

1. $[S^{\widetilde{\gamma}} : D^{\widetilde{\beta}}] = 1$.

2. $[S^{\widetilde{\delta}} : D^{\widetilde{\gamma}}] = 1$.

3. $[S^{\widetilde{\delta}} : D^{\widetilde{\beta}}] = 1$.

Using Schaper's Theorem, we can verify that

$$[S^{\widetilde{\alpha}} : D^{\langle 4\rangle_{\widetilde{B}}}] = [S^{\widetilde{\beta}} : D^{\langle 4\rangle_{\widetilde{B}}}] = [S^{\widetilde{\gamma}} : D^{\langle 4\rangle_{\widetilde{B}}}] = [S^{\widetilde{\delta}} : D^{\langle 4\rangle_{\widetilde{B}}}] = 0.$$

In this section we try to determine $[S^{\widetilde{\mu}} : D^{\widetilde{\lambda}}]$, where $\widetilde{\mu} \in \{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}\}$ and $\widetilde{\lambda} \in \{\langle 4,4\rangle_{\widetilde{B}}, \langle 4,5\rangle_{\widetilde{B}}\}$.

5.4.1 For $\widetilde{\lambda} = \langle 4,4\rangle_{\widetilde{B}}$

For $\widetilde{\lambda} = \langle 4,4\rangle_{\widetilde{B}}$, we know that $[S^{\beta} : D^{\lambda}] = [S^{\gamma} : D^{\lambda}] = 1$ and $[S^{\alpha} : D^{\lambda}] = [S^{\delta} : D^{\lambda}] = 0$, where $\lambda = \langle 5,5\rangle_B$. From (B1), we have $[S^{\widetilde{\alpha}}\uparrow^B : D^{\lambda}] = 4$. But then,

$$[S^{\widetilde{\alpha}}\uparrow^B : D^{\lambda}] = 2 \cdot [S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}] + [D^{\widetilde{\alpha}}\uparrow^B : D^{\lambda}].$$

Thus there are three cases to be considered:

1. $[S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}] = 0$ and $[D^{\widetilde{\alpha}}\uparrow^B : D^{\lambda}] = 4$. But this contradicts with (B2).

2. $[S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}] = 1$ and $[D^{\widetilde{\alpha}}\!\uparrow^{B} : D^{\lambda}] = 2$. From relationships (B1)-(B4), it is easy to see that $D^{\widetilde{\lambda}}$ is a composition factor of $S^{\widetilde{\alpha}}$ and $S^{\widetilde{\delta}}$ and is not composition factor of $S^{\widetilde{\beta}}$ and $S^{\widetilde{\gamma}}$. We may conclude that $[S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}] = [S^{\widetilde{\delta}} : D^{\widetilde{\lambda}}] = 1$ and $[S^{\widetilde{\beta}} : D^{\widetilde{\lambda}}] = [S^{\widetilde{\gamma}} : D^{\widetilde{\lambda}}] = 0$.

3. $[S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}] = 2$ and $[D^{\widetilde{\alpha}}\!\uparrow^{B} : D^{\lambda}] = 0$. Relationships (B1)-(B4) show that $D^{\widetilde{\lambda}}$ is a composition factor of $S^{\widetilde{\alpha}}$, $S^{\widetilde{\delta}}$, $S^{\widetilde{\beta}}$ and $S^{\widetilde{\gamma}}$. We also have $[S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}] = [S^{\widetilde{\delta}} : D^{\widetilde{\lambda}}] = 2$ and $[S^{\widetilde{\beta}} : D^{\widetilde{\lambda}}] = [S^{\widetilde{\gamma}} : D^{\widetilde{\lambda}}] = 1$.

Applying Schaper's Theorem yields

$$0 < [S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}] \leq 2 \qquad \text{and} \qquad 0 < [S^{\widetilde{\delta}} : D^{\widetilde{\lambda}}] \leq 2. \tag{5.3}$$

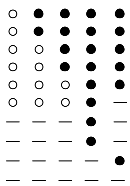
So, by Schaper's Theorem we cannot rule out any of the last two cases and we still have two possibilities:

$$\begin{aligned} [S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}] &= 2 \\ [S^{\widetilde{\beta}} : D^{\widetilde{\lambda}}] &= 1 \\ [S^{\widetilde{\gamma}} : D^{\widetilde{\lambda}}] &= 1 \\ [S^{\widetilde{\delta}} : D^{\widetilde{\lambda}}] &= 2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} [S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}] &= 1 \\ [S^{\widetilde{\beta}} : D^{\widetilde{\lambda}}] &= 0 \\ [S^{\widetilde{\gamma}} : D^{\widetilde{\lambda}}] &= 0 \\ [S^{\widetilde{\delta}} : D^{\widetilde{\lambda}}] &= 1. \end{aligned}$$

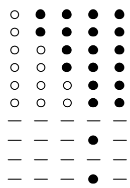
By repeated use of Theorem 4.3.8, we see that $[S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}]$ in $k\mathfrak{S}_{133}$ is equal to $[S^{\widehat{\alpha}} : D^{\widehat{\lambda}}]$, where $\widehat{\alpha} = (13, 8, 4)$ and $\widehat{\lambda} = (17, 8)$, in $k\mathfrak{S}_{25}$. In fact, we add 12 beads in the first 12 gaps of the abacus displays of $\widetilde{\alpha}$ and $\widetilde{\lambda}$ to get the abacus displays of $\widehat{\alpha}$ and $\widehat{\lambda}$. In the abacus displays below, we denote the additional beads by '$\circ$'.

$$\begin{array}{ccccc} - & \bullet & \bullet & \bullet & \bullet \\ - & \bullet & \bullet & \bullet & \bullet \\ - & - & \bullet & \bullet & \bullet \\ - & - & \bullet & \bullet & \bullet \\ - & - & - & \bullet & \bullet \\ - & - & - & \bullet & - \\ - & - & - & \bullet & - \\ - & - & - & \bullet & - \\ - & - & - & - & \bullet \\ - & - & - & - & - \end{array} \qquad\qquad \begin{array}{ccccc} - & \bullet & \bullet & \bullet & \bullet \\ - & \bullet & \bullet & \bullet & \bullet \\ - & - & \bullet & \bullet & \bullet \\ - & - & \bullet & \bullet & \bullet \\ - & - & - & \bullet & \bullet \\ - & - & - & \bullet & \bullet \\ - & - & - & - & - \\ - & - & - & \bullet & - \\ - & - & - & - & - \\ - & - & - & \bullet & - \end{array}$$

Abacus display of $\widetilde{\alpha}$ Abacus display of $\widetilde{\lambda}$



Abacus display of $\widehat{\alpha}$

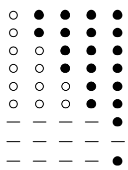


Abacus display of $\widehat{\lambda}$

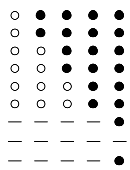
From the abacus displays above, it is clear that $\widehat{\lambda}$ has exactly 2 normal 4-nodes. By repeated use of Theorem 4.3.12, it is easy to see that

$$[S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}] = [S^{\widehat{\alpha}} : D^{\widehat{\lambda}}] \leq [S^{\breve{\alpha}} : D^{\breve{\lambda}}]. \tag{5.4}$$

where $\breve{\alpha} = \breve{\lambda} = (13, 4)$.



Abacus display of $\breve{\alpha}$



Abacus display of $\breve{\lambda}$

Since $\breve{\alpha} = \breve{\lambda}$, we have $[S^{\breve{\alpha}} : D^{\breve{\lambda}}] = 1$. So, by combining Equation 5.3 and Equation 5.4, we can conclude that $[S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}] = [S^{\widehat{\alpha}} : D^{\widehat{\lambda}}] = 1$.

Remark. As $\widehat{\alpha}$ also has exactly 2 removable beads, the summation in Equation 4.3 has only one factor, $[S^{\breve{\alpha}} : D^{\breve{\lambda}}]$. Alternatively, one can use Theorem 4.3.6 to obtain the same conclusion.

5.4.2 For $\widetilde{\lambda} = \langle 4, 5 \rangle_{\widetilde{B}}$

We know that $\lambda = \langle 5, 4 \rangle_B$ occurs once on S^{γ} and does not occur in other exceptional Specht modules.

From relationship (C1), D^{λ} is not a composition factor of $D^{\overline{\alpha}}\uparrow^{B}$. Using (C1)-(C6), we know that $D^{\overline{\lambda}}$ is a composition factor of $S^{\overline{\beta}}, S^{\overline{\gamma}}, S^{\overline{\kappa}}$ and not a composition factor of $S^{\overline{\alpha}}, S^{\overline{\delta}}, S^{\overline{\epsilon}}$. From (D2), we conclude that $D^{\widetilde{\lambda}}$ is not a composition factor of $S^{\widetilde{\beta}}$. Relationship (D1)-(D6) shows that $D^{\widetilde{\lambda}}$ is a composition factor of $S^{\widetilde{\alpha}}, S^{\widetilde{\gamma}}, S^{\widetilde{\delta}}$.

If $[S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}] = 2$ then, from (B1), at least two of $\{S^{\alpha}, S^{\beta}, S^{\gamma}\}$ have D^{λ} as their composition factors. Hence, we have a contradiction and we can conclude that $[S^{\widetilde{\alpha}} : D^{\widetilde{\lambda}}] = 1$. Similarly, we have $[S^{\widetilde{\gamma}} : D^{\widetilde{\lambda}}] = [S^{\widetilde{\delta}} : D^{\widetilde{\lambda}}] = 1$.

We give the complete decomposition matrix of the block of $k\mathfrak{S}_{133}$ in Appendix B.

Remark. Using Theorem 2.2.24 and the above results, we can calculate the dimension of Specht modules and their composition factors.

Appendix A

The Decomposition Matrix of $B(k\mathfrak{S}_{135})$

Please see the next page.

Appendix B

The Decomposition Matrix of $B(k\mathfrak{S}_{133})$

Please see the next page.

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